The Solution to the Partition Reconstruction Problem

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Definitions and Notation

- A *partition* λ of a positive integer n is an sequence $[\lambda_1, \lambda_2, \ldots, \lambda_m]$ of positive integers which satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\sum_{i=1}^{m} \lambda_i = n$.

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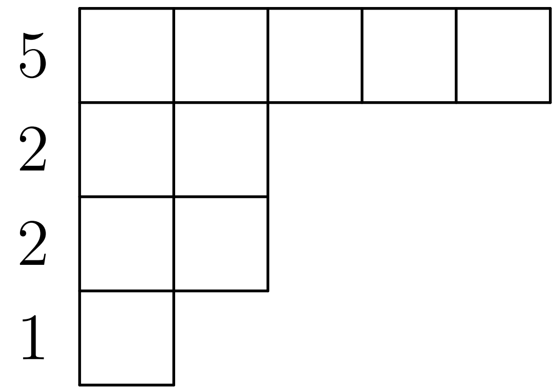
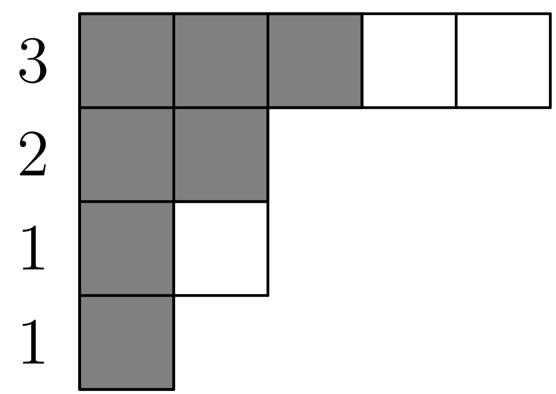
- $5 + 2 + 2 + 1 = 10$



Definitions and Notation

- Let λ be a partition of n, and let μ be a partition of $n-k$. Then μ is a k-*minor* of λ if $\mu_i \le \lambda_i$ for all i.
- $[3,2,1,1]$ is a 3-minor of $[5,2,2,1]$.



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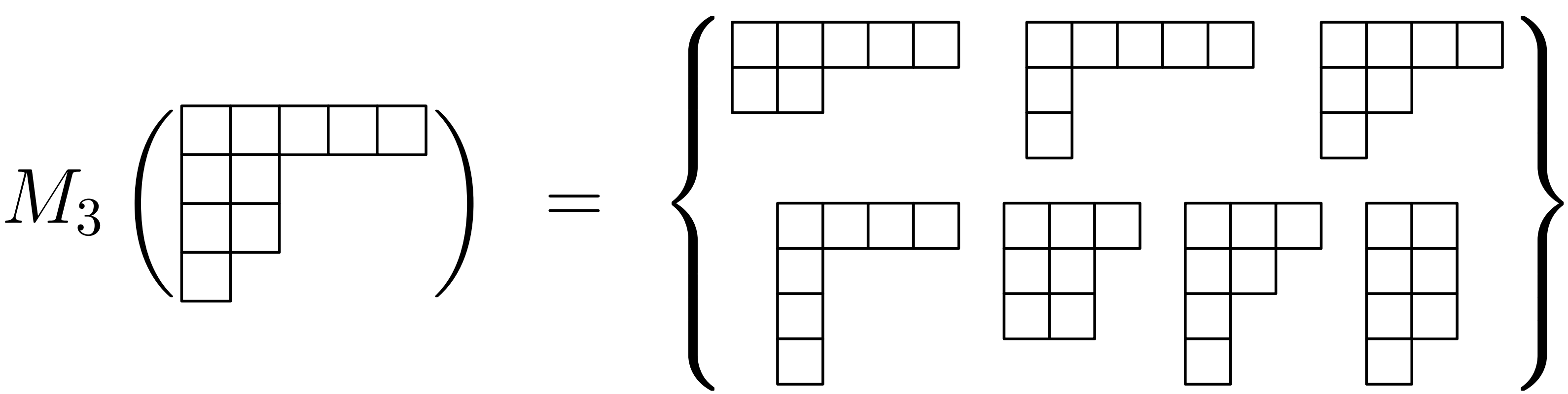
$$M_3([5,2,2,1]) = \{[5,2],[5,1,1],[4,2,1],[4,1,1,1],\\ [3,2,2],[3,2,1,1],[2,2,2,1]\}$$



The Problem

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- Formally, for which n and k does $M_k(\lambda) = M_k(\mu)$ imply $\lambda = \mu$ for any two partitions λ and μ of n?
- If this property holds, we say *reconstructibility holds*.

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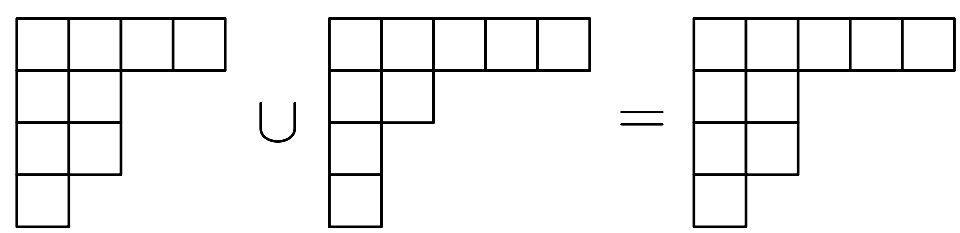

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- What is g?

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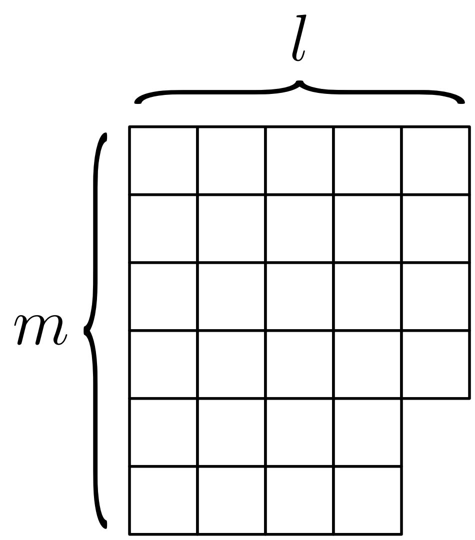
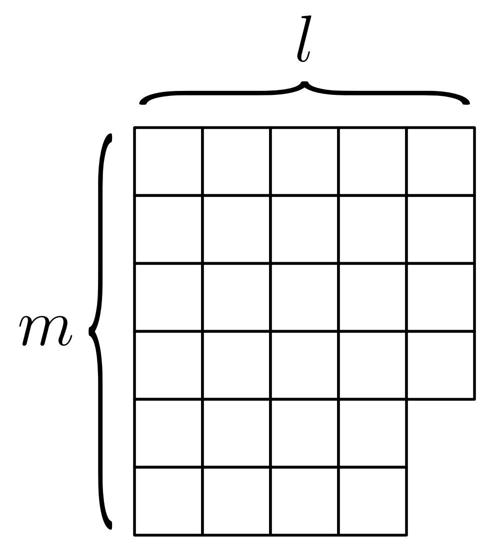
- If a partition of the form below appears for n, then $g(n) \leq m-2$.



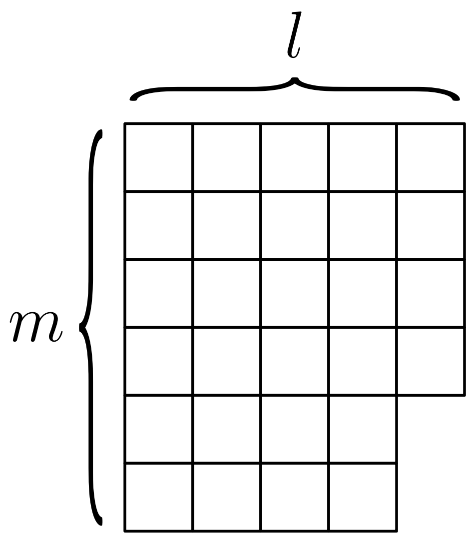
Properties of g

- Define $\rho(a)$ to be the smallest divisor d of a for which $d \geq \sqrt{a}$.



Properties of g

- Define $\rho(a)$ to be the smallest divisor d of a for which $d \geq \sqrt{a}$.
- We can show $g(n) \leq \rho(n+2) - 2$.



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Properties of g

- Define $\rho(a)$ to be the smallest divisor d of a for which $d \geq \sqrt{a}$.
- We can show $g(n) \leq \rho(n+2) - 2$.
- Recall that $g(n) \leq g(n-1) + 1$.
- In fact, the recursion

$$g(n) = \min\{g(n-1)+1, \rho(n+2)-2\}$$

holds for most positive integers n. Some counterexamples are: $n = 5, 12, 21, 32, \ldots$

Recursive formula for $g(n)$

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- Just $5, 12, 21, 32$. There are no other counterexamples!

Theorem. Let $n > 2$ be a positive integer other than 5, 12, 21, and 32. Then

$$g(n) = \min\{\rho(n+2) - 2, g(n-1) + 1\}.$$

Direct calculation shows that $g(2) = 0$, $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$.

Bounds on g

- We can use the recursion to obtain bounds on g:

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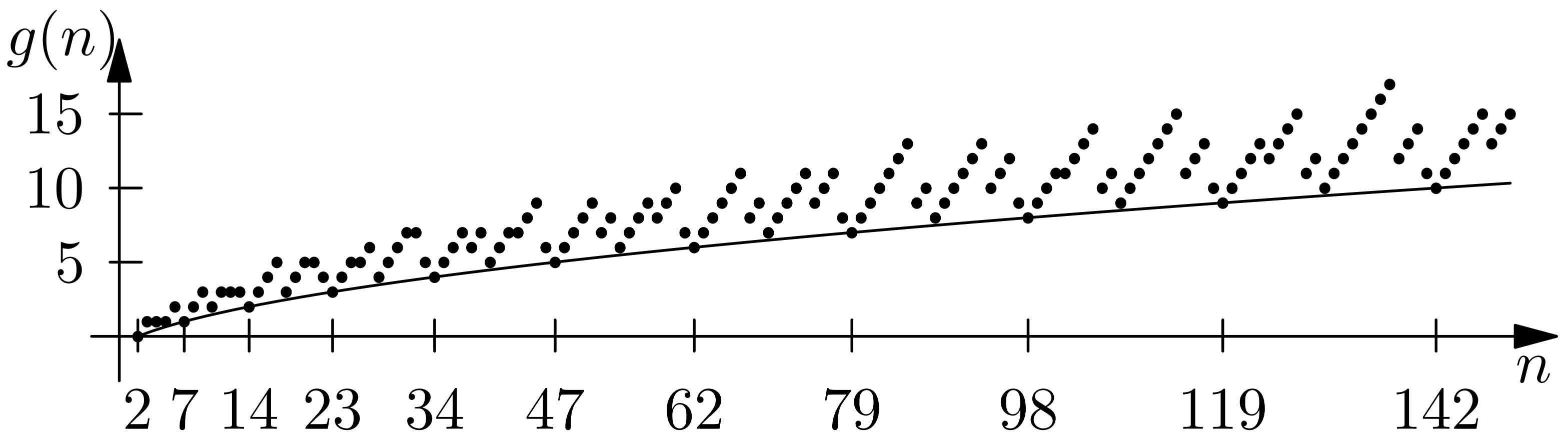
- Equality holds for the lower bound whenever $n+2$ is a perfect square.



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Theorem. Let n and k be positive integers with $k < n$. Then partitions of n can be reconstructed from their sets of k-minors if and only if $k \leq g(n)$, where

$$g(n) = \min_{0 \leq t \leq n} \rho(n + 2 - t) - 2 + t$$

for $n \notin \{5, 12, 21, 32\}$, and $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$.

Applications

- The partition reconstruction problem was posed by J. Siemons and O. Pretzel, in an attempt to answer the following:
  For which n and k can we always reconstruct an irreducible character of S_n from the irreducible components of its restriction to the stabilizer of $\{1, 2, \ldots, k\}$?

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  For which n and k can we always reconstruct an irreducible character of S_n from the irreducible components of its restriction to the stabilizer of $\{1, 2, \ldots, k\}$?
- Our main theorem also completely answers this question: we can do so if and only if $k \leq g(n)$.

Acknowledgments

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