Crystallizing the Schur Q-functions

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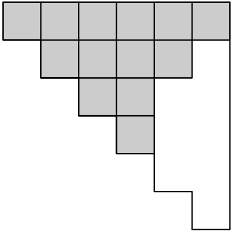
AMS Fall Western Sectional
Nov 4, 2017

Shifted tableaux

- **Shifted partitions:** Partitions with distinct parts; *i*th row shifted to the right *i* steps.



$\lambda = (6, 4, 2, 1)$

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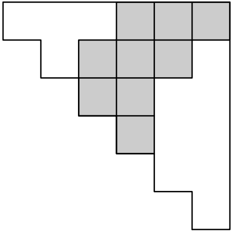

$$\lambda = (6, 4, 2, 1)$$
$$\mu = (3, 1)$$

- **Skew shape:** λ/μ

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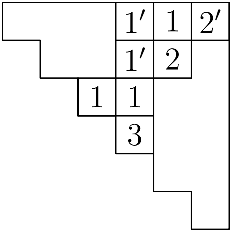

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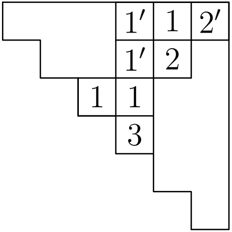

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- **Canonical form:** First i or i' is always *unprimed* in reading order (read rows from bottom to top, $3111'21'12'$).

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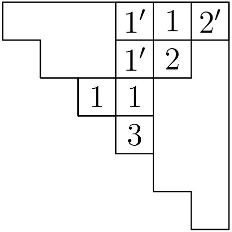

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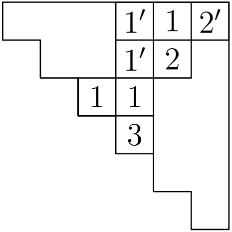

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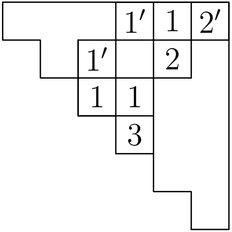

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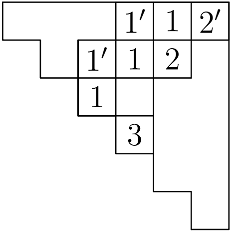

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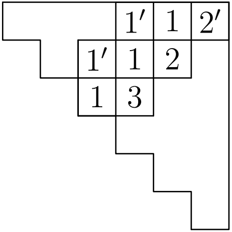

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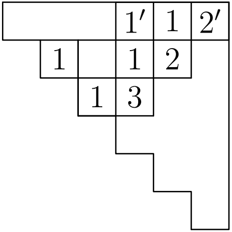

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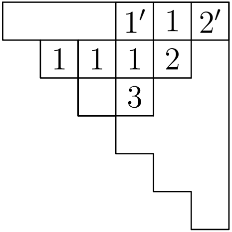

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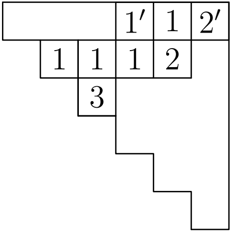

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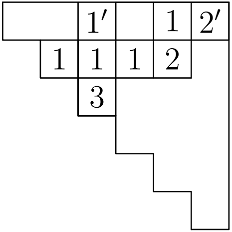

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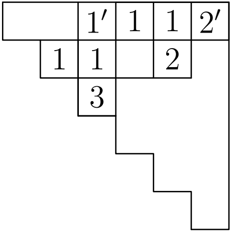

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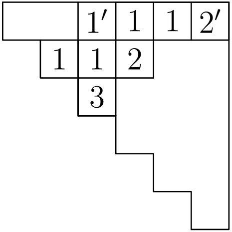

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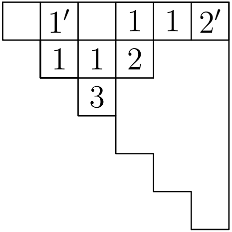

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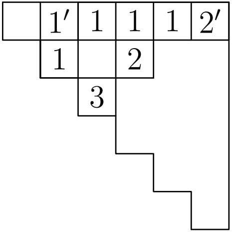

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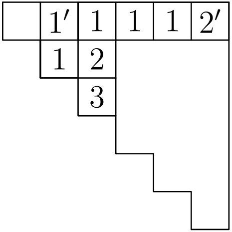

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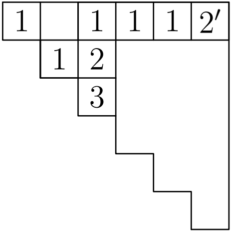

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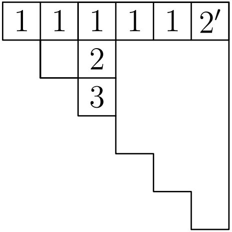

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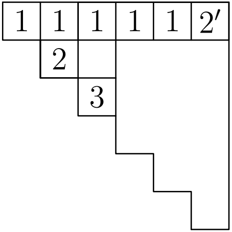

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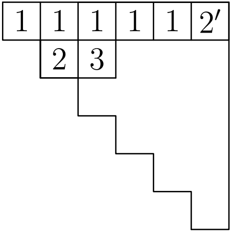

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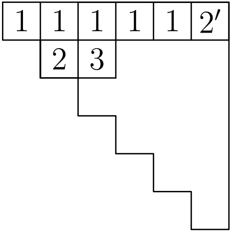
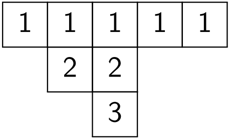
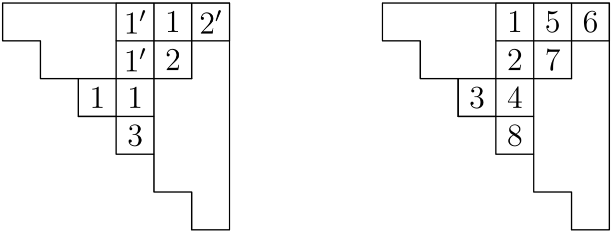


- **Highest weight**: Rectifies to shifted tableau with all *i*'s in *i*th row:



Standardization

- **Standardization order:** Break ties by reading order for unprimed entries, reverse reading order for primed entries



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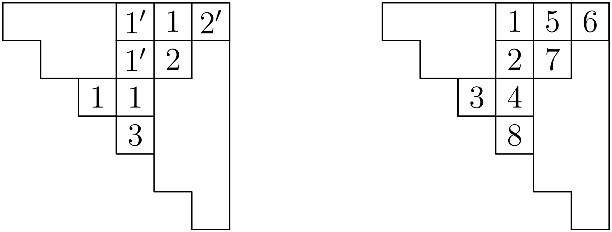

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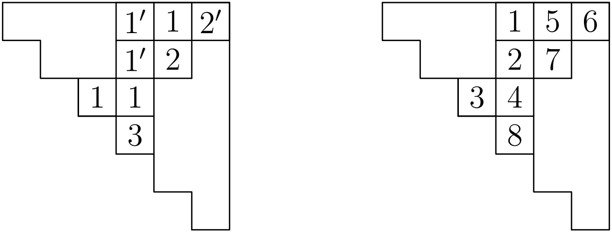

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- **Monomial weight:** $\mathbf{x}^{wt(T)} = x_1^{m_1} x_2^{m_2} \cdots$. Above, $x_1^5 x_2^2 x_3$.

Shifted Littlewood-Richardson Rule

- **Schur Q-functions:** Let $\ell(\mathrm{wt}\,T)$ be the number of nonzero entries in $\mathrm{wt}\,T$.

$$Q_{\lambda/\mu}(x_1, x_2, \ldots) = \sum_{T \in \mathrm{ShST}(\lambda/\mu)} 2^{\ell(\mathrm{wt}\,T)} \mathbf{x}^{\mathrm{wt}\,T}$$

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$$Q_{\lambda/\mu} = \sum f^{\lambda}_{\mu\nu} Q_{\nu}$$

where $f^{\lambda}_{\mu\nu}$ is the number of **highest weight** canonical shifted semistandard tableaux of shape λ/μ and weight ν.

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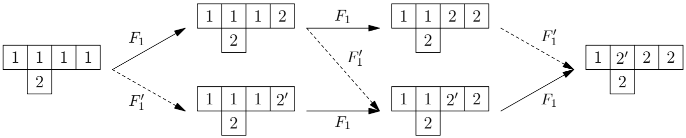
Straight shapes, two letters

- Restrict to alphabet $\{1', 1, 2', 2\}$. Shape can have two rows:



Or one row:

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array} \overset{F_1}{\underset{F_1'}{\rightrightarrows}} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline\end{array} \overset{F_1}{\underset{F_1'}{\rightrightarrows}} \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline\end{array} \overset{F_1}{\underset{F_1'}{\rightrightarrows}} \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline\end{array} \overset{F_1}{\underset{F_1'}{\rightrightarrows}} \begin{array}{|c|c|c|c|}\hline 2&2&2&2\\\hline\end{array} \overset{F_1}{\underset{F_1'}{\rightrightarrows}} \emptyset$$

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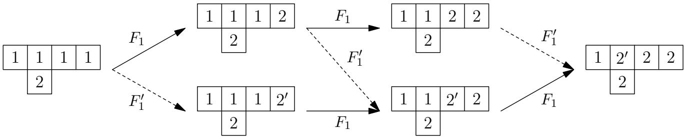

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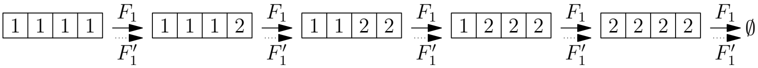

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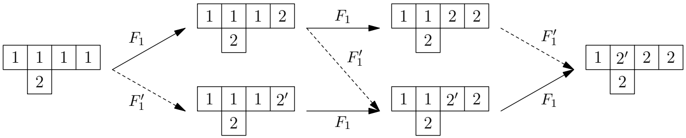

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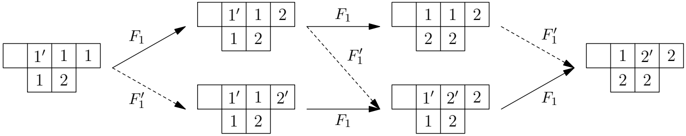

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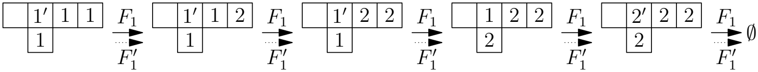

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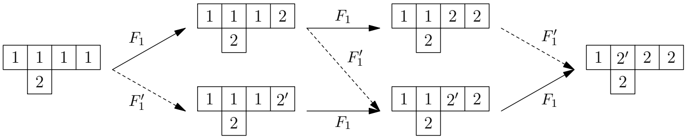

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- Need two operators F_1, F_1' and their partial inverses E_1, E_1'.
- **Coplacticity**: Extend to skew shapes by applying outer slides.
- **General operators:** F_i, F_i', E_i, E_i' act on the strip of $i', i, (i+1)', i+1$ letters, by JDT rectifying, applying the appropriate operator, and unrectifying.

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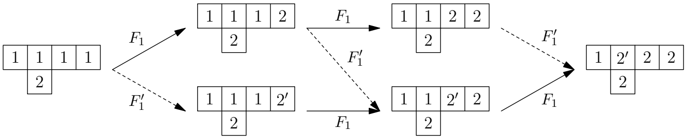

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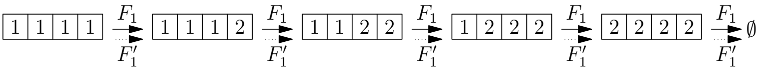

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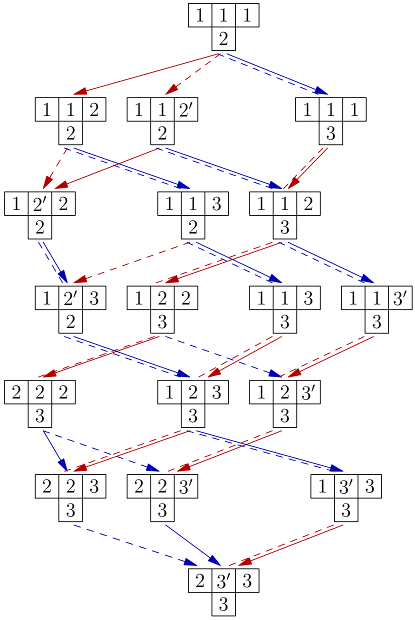
- **General operators:** F_i, F_i', E_i, E_i' act on the strip of $i', i, (i+1)', i+1$ letters, by JDT rectifying, applying the appropriate operator, and unrectifying.
- Highest weight iff killed by all raising operators E_i, E_i'.

Crystal-like structure

- “Crystal graph” for $i = 1, 2$:

$$\boxed{\color{red}{F_1} \quad F_1' \qquad \color{blue}{F_2} \quad F_2'}$$



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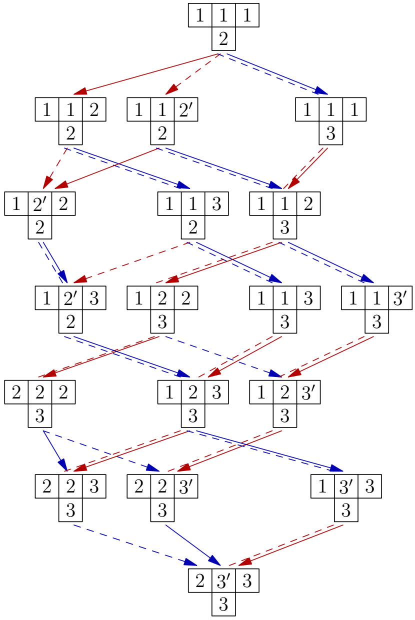
- Characters are Schur Q-functions:

$$\sum_{T \text{ in crystal}} 2^{\ell(wt(T))} x^{\mathrm{wt}\, T} = Q_\lambda$$

Graph structure implies Q_λ is symmetric.



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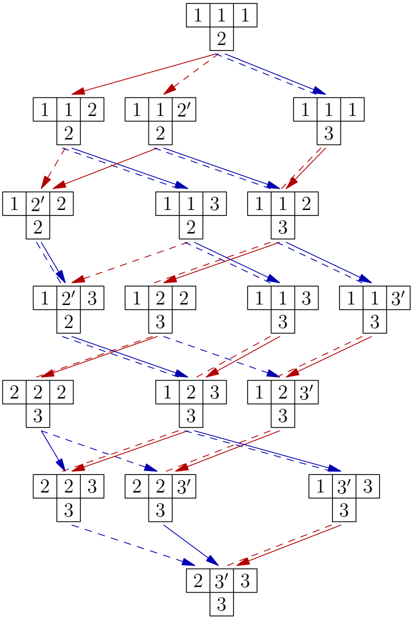
- Connected components for skew shapes give LR rule for $Q_{\lambda/\mu}$.

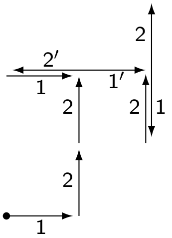


Lattice walks of words

- **Walk** of $w = w_1 w_2 \cdots w_n \in \{1', 1, 2', 2\}^n$ is a lattice walk in first quadrant from $(x_0, y_0) = (0, 0)$ to (x_n, y_n), with w_i labeling the step $(x_i, y_i) \to (x_{i+1}, y_{i+1})$. Directions:

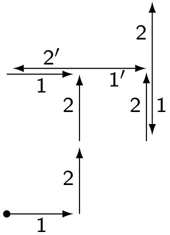
$\xrightarrow{1'}$	$\xrightarrow{1}$	$\uparrow 2'$	$\uparrow 2$	if $x_i y_i = 0$
$\xrightarrow{1'}$	$\downarrow 1$	$\xleftarrow{2'}$	$\uparrow 2$	if $x_i y_i \neq 0$

- **Example:** The walk of $1222'11'122$ looks like:



Properties of lattice walks (G., Levinson, Purbhoo)

- **Rectification:** Endpoint (x_n, y_n) tells much about $\mathrm{rect}(w)$:



1	1	1	1	2	2	2
	2	2				

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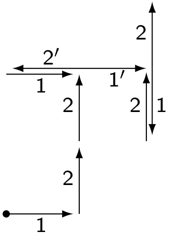

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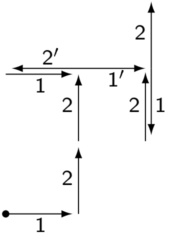

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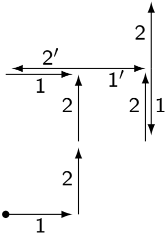

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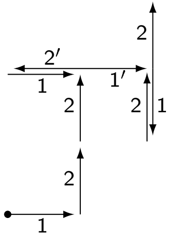

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- **Highest weight:** A word w with letters in $\{1', 1, 2', 2\}$ has $E_1(w) = E_1'(w) = \varnothing$ iff its walk ends on the x-axis.
- Proofs via **Knuth equivalence:** An elementary shifted Knuth move (Sagan, Worley) does not change the endpoint of the walk.

The operation F_1 on words

- Let $w \in \{1', 1, 2', 2\}^n$. An **F-critical substring** of w is a substring of any of the types and locations below.

Type	Substring	Starting Location	Transformation
1F	$1(1')^*2'$	$y = 0$ or $y = 1,\ x \geqslant 1$	$2'(1')^*2$
2F	$1(2)^*1'$	$x = 0$ or $x = 1,\ y \geqslant 1$	$2'(2)^*1$
3F	1	$y = 0$	2
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- If no F-critical substrings, $F_1(w) = \varnothing$.

Example

Type	Substring	Starting Location	Transformation
1F	$1(1')^*2'$	$y=0$ or $y=1,\ x \geqslant 1$	$2'(1')^*2$
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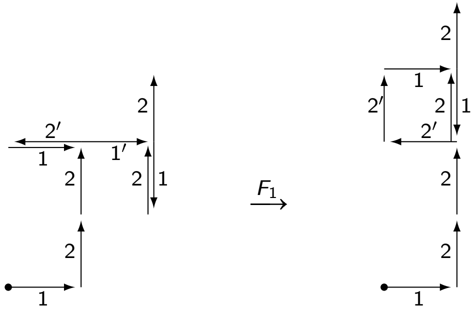
The word $w = 1222'11'122$ has a type 2F substring at $11'$, and this is its final F-critical substring. Thus $F_1(w) = 1222'2'1122$.



The operation E_1 on words

- Let $w \in \{1', 1, 2', 2\}^n$. An **E-critical substring** of w is a substring of any of the types and locations below.

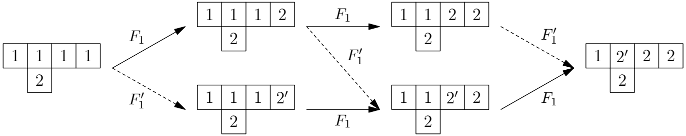
Type	Substring	Starting Location	Transformation
1E	$2'(2)^*1$	$x = 0$ or $x = 1,\ y \geqslant 1$	$1(2)^*1'$
2E	$2'(1')^*2$	$y = 0$ or $y = 1,\ x \geqslant 1$	$1(1')^*2'$
3E	$2'$	$y = 0$	$1'$
4E	2	$x = 0$	1
5E	1 or $2'$	$y = 1,\ x \geqslant 1$	$\varnothing$

- **Final substring** is the E-critical substring $w_i \cdots w_j$ with largest i, breaking ties by largest j.
- $E_1(w)$ defined by applying the appropriate transformation to the final E-critical substring of w.
- If there are no E-critical substrings we define $E_1(w) = \varnothing$.

Properties of E_1 and F_1 (G., Levinson, Purbhoo.)

Theorem. The operators E_1 and F_1 are:

- **Defined on tableaux:** Applying E_1 or F_1 to the reading word of a skew shifted semistandard tableau preserves semistandardness of the entries.
- **Agree with diagram on straight shapes:**



- **Coplactic:** E_1 and F_1 commute with all sequences of inner or outer JDT slides. (Difficult!)
- **Partial inverses:** $E_1(T) = T'$ if and only if $F_1(T') = T$.

Primed operators E_1' and F_1'

- $E_1'(w)$ is defined by changing the last $2'$ in w to a 1 if this does not change the standardization. Otherwise $E_1'(w) = \varnothing$.
- $F_1'(w)$ is defined by changing the last 1 in w to a $2'$ if this does not change the standardization. Otherwise $F_1'(w) = \varnothing$.
- Two maximal F_1' chains:

$$12211' \xrightarrow{F_1'} 1222'1' \xrightarrow{F_1'} \varnothing$$

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- **Theorem.** The operations E_1' and F_1' are:
    - Coplactic and well-defined on skew shifted tableaux.
    - Partial inverses: if $E_1'(T) = T'$ then $F_1'(T') = T$.
    - Have chains of length 2 unless the rectification shape has one row; in the latter case they coincide with E_1 and F_1.

Properties

- E'_1, E_1, F'_1, F_1 all commute with each other.

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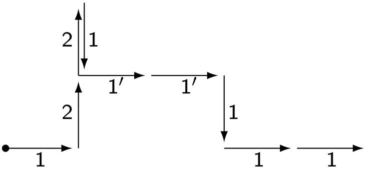

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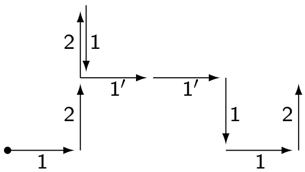

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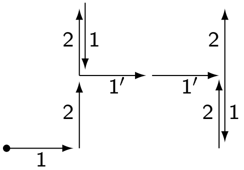

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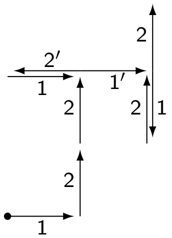

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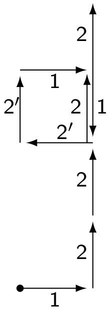

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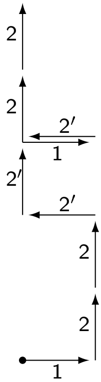

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- **Schubert curves:** certain 1-dimensional intersections of Schubert varieties

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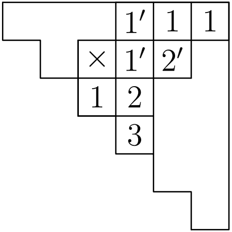

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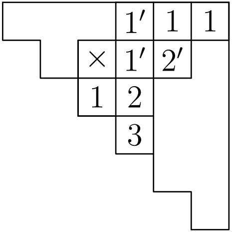

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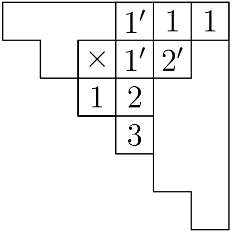

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- Operators F_i, F_i' give us a new easier rule that avoids rectification!

Thank You!

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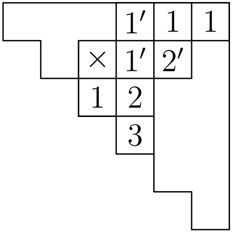

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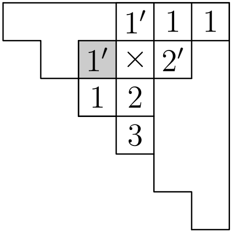

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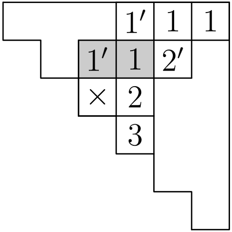

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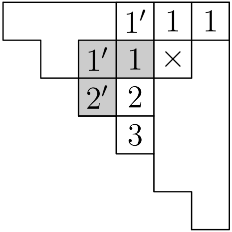

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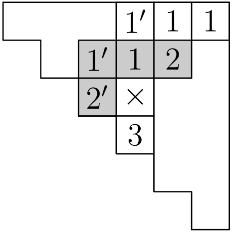

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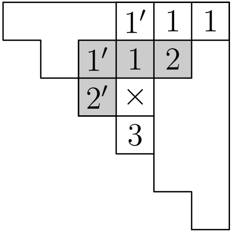

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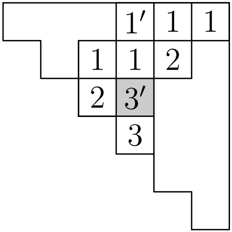

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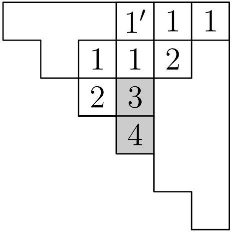

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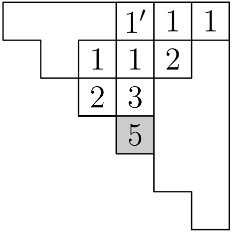

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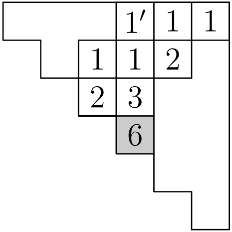

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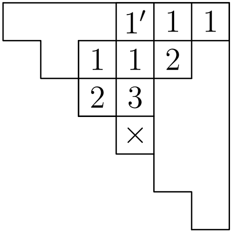

Larger Phase 2 example

						1
					1	2
			×	3		
	1	1				
	2					
	3					
2	4					

Larger Phase 2 example

$$
\begin{array}{ccccccc}
 & & & & & & \boxed{1} \\
 & & & & & \boxed{1} & \boxed{2} \\
 & & & \boxed{1'} & \boxed{3} & & \\
 & \boxed{1} & \boxed{1} & & & & \\
 & \boxed{2} & & & & & \\
 & \boxed{3} & & & & & \\
\boxed{2} & \boxed{4} & & & & &
\end{array}
$$

Larger Phase 2 example

$$
\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & 1 & 2 \\
 & & & 1 & 3 & & \\
 & 1 & 2' & & & & \\
 & 2 & & & & & \\
 & 3 & & & & & \\
2 & 4 & & & & &
\end{array}
$$

Larger Phase 2 example

$$
\begin{array}{ccccccc}
 & & & & & & \boxed{1} \\
 & & & & & \boxed{1} & \boxed{2} \\
 & & & \boxed{1} & \boxed{3} & & \\
 & \boxed{1} & \boxed{2} & & & & \\
 & \boxed{3'} & & & & & \\
 & \boxed{3} & & & & & \\
\boxed{2} & \boxed{4} & & & & &
\end{array}
$$

Larger Phase 2 example

$$
\begin{array}{ccccccc}
 & & & & & & \boxed{1} \\
 & & & & & \boxed{1} & \boxed{2} \\
 & & & \boxed{1} & \boxed{3} & & \\
 & \boxed{1} & \boxed{2} & & & & \\
 & \boxed{3} & & & & & \\
 & \boxed{4'} & & & & & \\
\boxed{2} & \boxed{4} & & & & &
\end{array}
$$

Larger Phase 2 example

						1
					1	2
			1	3		
	1	2				
	3					
	4					
2	5					

Larger Phase 2 example

						1
					1	2
			1	3		
	1	2				
	3					
	4					
2	6					

Larger Phase 2 example

						1
					1	2
			1	3		
	1	2				
	3					
	4					
2	7					

Larger Phase 2 example

						1
					1	2
			1	3		
	1	2				
	3					
	4					
2	$\times$					

Application: Schubert curves

- **Grassmannian**: $\mathrm{Gr}(n, k)$ is the variety of k-dimensional subspaces of $\mathbb{C}^n$.

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Application: Schubert curves

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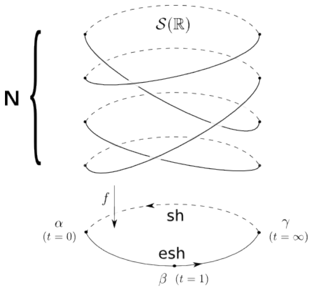
- **Schubert curve:** A one-dimensional intersection of Schubert varieties.
- Special Schubert curves: Three partitions α, β, γ with $|\alpha| + |\beta| + |\gamma| = k(n-k) - 1$. Define

$$\boxed{S = S(\alpha, \beta, \gamma) = \Omega_\alpha(\mathcal{F}_0) \cap \Omega_\beta(\mathcal{F}_1) \cap \Omega_\gamma(\mathcal{F}_\infty)}$$

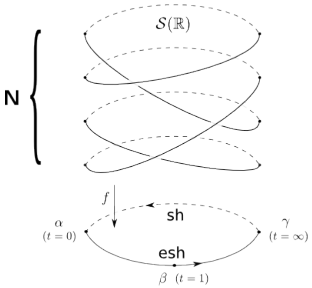
where the flag $\mathcal{F}_t$ is the maximally tangent flag at t of the *rational normal curve* in $\mathbb{P}^{n-1}$:

$$(1 : t) \mapsto (1 : t : t^2 : \cdots : t^{n-1})$$

Real geometry of S

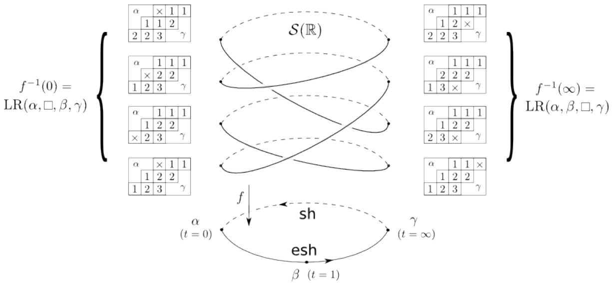


Real geometry of S



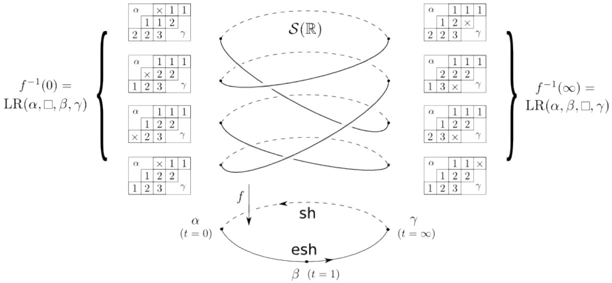
Theorem(s). (Levinson, Speyer.) There is a degree-N map $f : S \to \mathbb{P}^1$ that makes $S(\mathbb{R})$ a smooth covering of the circle $\mathbb{RP}^1$, with finite fibers of size $N = c^{\ydiagram{3,3}}_{\alpha,\square,\beta,\gamma}$.

Real geometry of S



- (Fiber over 0) $\leftrightarrow$ Tableaux of shape γ^c/α with one **inner** corner $\times$ and the rest a highest weight tableau of weight β.
- (Fiber over ∞) $\leftrightarrow$ Tableaux of shape γ^c/α with one **outer** corner $\times$ and the rest a highest weight tableau of weight β.

Real geometry of S

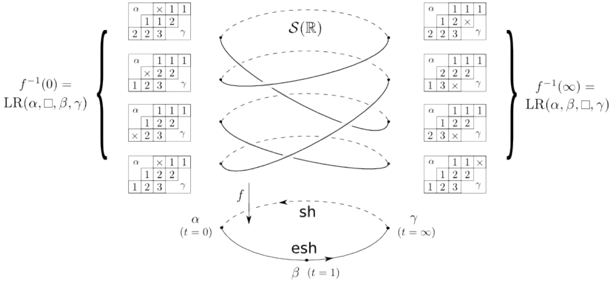


- The arcs of $S(\mathbb{R})$ covering $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively induce the *shuffling* and *evacuation-shuffling* bijections sh and esh:

$$\mathrm{LR}(\alpha, \square, \beta, \gamma) \underset{\mathrm{sh}}{\overset{\mathrm{esh}}{\rightleftarrows}} \mathrm{LR}(\alpha, \beta, \square, \gamma)$$

- **Monodromy operator:** $\omega = \mathrm{sh} \circ \mathrm{esh}$. Cycles of ω correspond to connected components of $S(\mathbb{R})$.

Real geometry of S



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Shuffling

- **Shuffling**, or **JDT**: Do an outer jeu de taquin slide with the $\boxtimes$ as the empty square to get an element of $\mathrm{LR}(\alpha, \square, \beta, \gamma)$.

α			1	1	1
		1	2	2	2
	2	3	3		
1	3	4	4	γ	
3	4	5	$\times$		

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		1	2	2	2
	2	3	3		
1	3	4	4		
3	4	$\times$	5		

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			1	1	1
		1	2	2	2
	2	3	3		
1	3	$\times$	4		
3	4	4	5		

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			1	1	1
		1	2	2	2
	2	$\times$	3		
1	3	3	4		
3	4	4	5		

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	$\times$	2	3		
1	3	3	4		
3	4	4	5		

Evacuation-shuffling

- Conjugation of shuffling by JDT rectification.
- **Rectification:** Treat $\times$ as 0.

		1	1	1
	$\times$	2	2	
1	2	3		

T

		1	1	1
	$\times$	2	2	
1	2	3		

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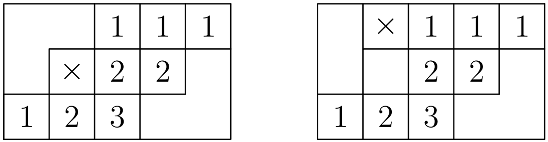

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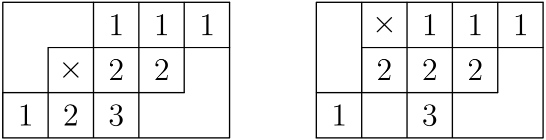

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	2	2	2	
1	3			

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	$\times$	2	2	
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	2	2	2	
1	3	1		

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	$\times$	2	2	
1	2	3		

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	$\times$	1	1	1
1	2	2	2	
	3	1		

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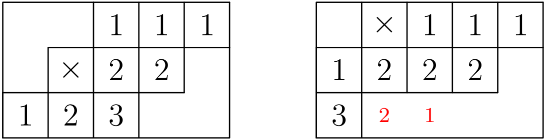

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		1	1	1
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1	2	3		

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$\times$		1	1	1
1	2	2	2	
3	2	1		

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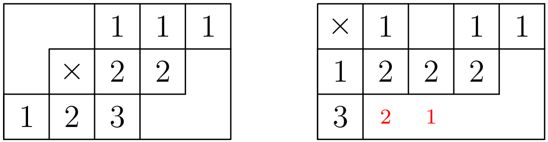

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		1	1	1
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$\times$	1	1		1
1	2	2	2	
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$\times$	1	1	1	3
1	2	2	2	
3	2	1		

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		1	1	1
	$\times$	2	2	
1	2	3		

T

1	1	1	1	3
$\times$	2	2	2	
3	2	1		

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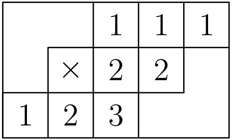

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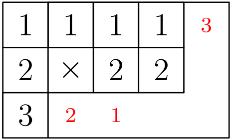

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		1	1	1
	$\times$	2	2	
1	2	3		

T

1	1	1	1	3
2	2	$\times$	2	
3	2	1		

Evacuation-shuffling

- Conjugation of shuffling by JDT rectification.
- **Rectification:** Treat $\times$ as 0.
- **Shuffling**
- **Un-rectification:** Treat $\times$ as largest entry.

		1	1	1
	$\times$	2	2	
1	2	3		

T

1	1	1	1	3
2	2	2	$\times$	
3	2	1		

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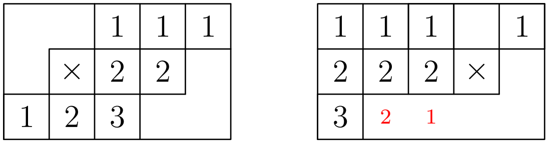

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		1	1	1
	$\times$	2	2	
1	2	3		

T

1	1		1	1
2	2	2	$\times$	
3	2	1		

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		1	1	1
	$\times$	2	2	
1	2	3		

T

1		1	1	1
2	2	2	$\times$	
3	2	1		

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		1	1	1
	$\times$	2	2	
1	2	3		

T

	1	1	1	1
2	2	2	$\times$	
3	2	1		

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	$\times$	2	2	
1	2	3		

T

	1	1	1	1
2	2	2	$\times$	
	3	1		

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		1	1	1
	$\times$	2	2	
1	2	3		

T

	1	1	1	1
	2	2	$\times$	
2	3	1		

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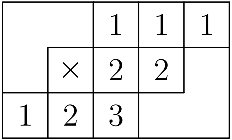

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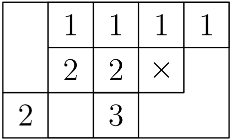

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		1	1	1
	$\times$	2	2	
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T

	1	1	1	1
		2	$\times$	
2	2	3		

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		1	1	1
	$\times$	2	2	
1	2	3		

T

		1	1	1
	1	2	$\times$	
2	2	3		

$\mathrm{esh}(T)$

Evacuation-shuffling

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		1	1	1
	$\times$	2	2	
1	2	3		

T

		1	1	1
	1	2	$\times$	
2	2	3		

$\mathrm{esh}(T)$

- **Shuffle again** to compute $\omega = \mathrm{sh} \circ \mathrm{esh}$:

ωT:

		$\times$	1	1
	1	1	2	
2	2	3		

Local rule for evacuation-shuffling

- Recall: esh consists of rectifying, shuffling, and un-rectifying.
- (G., Levinson.) Local rule, without rectifying: Start at $i = 1$.

						1	1	1
			$\times$	1	1	2	2	
			1	2	2	3		
			3	3	4			
		4	4					
2	3	5						

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						1	1	1
			1	1	1	2	2	
			$\times$	2	2	3		
			3	3	4			
		4	4					
2	3	5						

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			3	3	4			
		4	4					
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$\times$	3	5						

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			1	1	1	2	2	
			2	2	2	3		
			3	3	4			
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3	3	5						

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						1	1	1
			1	1	1	2	2	
			2	2	2	3		
			3	3	4			
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  - **Phase 2.** Replace the $\boxtimes$ with i and apply $F_i, F_{i+1}, \ldots$ in that order until only one entry is changing. Then replace that entry with $\boxtimes$.

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 2 & 2 & \\
 & & & 2 & 2 & 2 & 3 & & \\
 & & & 3 & 3 & 5 & & & \\
 & & 4 & 4 & & & & & \\
3 & 4 & 5 & & & & & &
\end{array}
$$

Local rule for evacuation-shuffling

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						1	1	1
			1	1	1	2	2	
			2	2	2	3		
			3	3	6			
		4	4					
3	4	5						

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						1	1	1
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			2	2	2	3		
			3	3	7			
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 & & & & & & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 2 & 2 & \\
 & & & 2 & 2 & 2 & 3 & & \\
 & & & 3 & 3 & \times & & & \\
 & & 4 & 4 & & & & & \\
3 & 4 & 5 & & & & & &
\end{array}
$$

Geometric consequences (G., Levinson)

- **Connections to K-theory:** Pechenik and Yong's "genomic tableaux" that are used to compute in $K(\mathrm{Gr}(n,k))$ appear naturally as steps in the algorithm.

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- Schubert curves can have **arbitrarily high arithmetic genus** (connected ω-orbits with many genomic tableaux appearing).
- Schubert curves can have **arbitrarily many connected components**, and in fact can be a disjoint union of arbitrarily many copies of $\mathbb{P}^1$ (when all tableaux of the given shape and content are fixed by ω).