A combinatorial approach to Macdonald q, t-symmetry via the Carlitz bijection

Maria Monks Gillespie
UC Davis

FPSAC Vancouver
July 4, 2016

Background on Macdonald polynomials

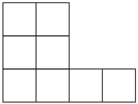
- **Macdonald polynomials** $\widetilde{H}_\mu(X; q, t)$ are symmetric functions in $X = \{x_1, x_2, x_3, \ldots\}$ with coefficients in $\mathbb{Z}[q, t]$.

Background on Macdonald polynomials

- **Macdonald polynomials** $\widetilde{H}_\mu(X; q, t)$ are symmetric functions in $X = \{x_1, x_2, x_3, \ldots\}$ with coefficients in $\mathbb{Z}[q, t]$.
- Here $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}_+^k$ is any **partition**, i.e. $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.

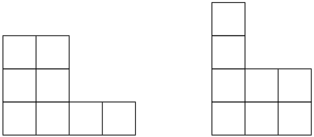
Background on Macdonald polynomials

- **Macdonald polynomials** $\widetilde{H}_\mu(X; q, t)$ are symmetric functions in $X = \{x_1, x_2, x_3, \ldots\}$ with coefficients in $\mathbb{Z}[q, t]$.
- Here $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}_+^k$ is any **partition**, i.e. $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.
- Young diagram: $\mu = (4, 2, 2)$



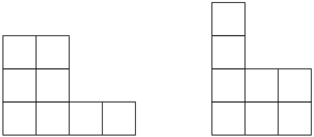
Background on Macdonald polynomials

- **Macdonald polynomials** $\widetilde{H}_\mu(X; q, t)$ are symmetric functions in $X = \{x_1, x_2, x_3, \ldots\}$ with coefficients in $\mathbb{Z}[q, t]$.
- Here $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}_+^k$ is any **partition**, i.e. $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.
- Young diagram: $\mu = (4, 2, 2)$, $\mu^* = (3, 3, 1, 1)$



Background on Macdonald polynomials

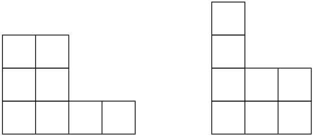
- **Macdonald polynomials** $\widetilde{H}_\mu(X; q, t)$ are symmetric functions in $X = \{x_1, x_2, x_3, \dots\}$ with coefficients in $\mathbb{Z}[q, t]$.
- Here $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{Z}_+^k$ is any **partition**, i.e. $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.
- Young diagram: $\mu = (4, 2, 2)$, $\mu^* = (3, 3, 1, 1)$



- Related to classical Macdonald polynomials P_λ by a transformation, arise naturally in the geometry of the Hilbert scheme of points in the plane. (Haiman)

Background on Macdonald polynomials

- **Macdonald polynomials** $\widetilde{H}_\mu(X; q, t)$ are symmetric functions in $X = \{x_1, x_2, x_3, \ldots\}$ with coefficients in $\mathbb{Z}[q, t]$.
- Here $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}_+^k$ is any **partition**, i.e. $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.
- Young diagram: $\mu = (4, 2, 2)$, $\mu^* = (3, 3, 1, 1)$



- Related to classical Macdonald polynomials P_λ by a transformation, arise naturally in the geometry of the Hilbert scheme of points in the plane. (Haiman)
- q, t-symmetry (via geometry): $\widetilde{H}_\mu(X; q, t) = \widetilde{H}_{\mu^*}(X; t, q)$

A combinatorial formula (Haglund, Haiman, Loehr)

- **Combinatorial Formula:**

$$\widetilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$$

A combinatorial formula (Haglund, Haiman, Loehr)

- **Combinatorial Formula:**

$$\widetilde{H}_{\mu}(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^{\sigma}$$

- $\sigma : \mu \to \mathbb{Z}_+$ is a filling of the Young diagram of μ with positive integers.

Example: $\mu = (3,2,2)$, $\sigma = \begin{array}{|c|c|c} \hline 5 & 1 & \\ \hline 2 & 1 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$

A combinatorial formula (Haglund, Haiman, Loehr)

- **Combinatorial Formula:**

$$\widetilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$$

- $\sigma : \mu \to \mathbb{Z}_+$ is a filling of the Young diagram of μ with positive integers.

  Example: $\mu = (3,2,2)$, $\sigma =$

5	1	
2	1	
2	3	2

- $x^\sigma = \prod x_i^{|\sigma^{-1}(i)|}$, in this case $x_1^2 x_2^3 x_3 x_5$

A combinatorial formula (Haglund, Haiman, Loehr)

- **Combinatorial Formula:**

$$\widetilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$$

- $\sigma : \mu \to \mathbb{Z}_+$ is a filling of the Young diagram of μ with positive integers.

  Example: $\mu = (3,2,2)$, $\sigma = $

5	1	
2	1	
2	3	2

- $x^\sigma = \prod x_i^{|\sigma^{-1}(i)|}$, in this case $x_1^2 x_2^3 x_3 x_5$
- inv and maj are statistics on fillings that generalize inv and maj on permutations.

A combinatorial formula (Haglund, Haiman, Loehr)

- **Combinatorial Formula:**

$$\widetilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$$

- $\sigma:\mu\to\mathbb{Z}_+$ is a filling of the Young diagram of μ with positive integers.

  Example: $\mu = (3,2,2)$, $\sigma =$

5	1	
2	1	
2	3	2

- $x^\sigma = \prod x_i^{|\sigma^{-1}(i)|}$, in this case $x_1^2 x_2^3 x_3 x_5$
- inv and maj are statistics on fillings that generalize inv and maj on permutations.
- q,t-symmetry not obvious from this formula

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) =$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) =$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) =$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: inv(51423) =

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) =$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51{\color{red}42}3) =$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\text{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\text{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\text{inv}(51423) =$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\text{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\text{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\text{inv}(51423) = 6$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: inv(51423) = 6 , maj(51423) =

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) = 6$, $\mathrm{maj}(\mathbf{5}1423) = 1 +$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\text{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\text{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\text{inv}(51423) = 6$, $\text{maj}(51\mathbf{4}23) = 1 + 3$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) = 6$, $\mathrm{maj}(51423) = 1 + 3 = 4$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) = 6$, $\mathrm{maj}(51423) = 1 + 3 = 4$
- inv and maj are equidistributed:

$$\sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)} = (1)(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

Background: inv and maj on permutations

- Let $\pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. Then

$$\mathrm{inv}(\pi) = |\{(i,j) : i < j, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d$$

- Example: $\mathrm{inv}(51423) = 6$, $\mathrm{maj}(51423) = 1 + 3 = 4$
- inv and maj are equidistributed:

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = (1)(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

- Combinatorial proofs: Find a "weight-preserving" bijection $\phi : S_n \to S_n$, i.e. a bijection such that $\mathrm{maj}(\phi(\pi)) = \mathrm{inv}(\pi)$. Several such maps have been found (Carlitz, Foata,...)

The Carlitz bijection $S_n \to S_n$

- **Carlitz codes:** Let $C_n = \{c_1 c_2 \cdots c_n : \forall i, 0 \leq c_i \leq n - i\}$. Define the *weight* of a code $c \in C_n$ to be $|c| = \sum_i c_i$. Then

$$\sum_{c \in C_n} q^{|c|} = (1)(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

The Carlitz bijection $S_n \to S_n$

- **Carlitz codes:** Let $C_n = \{c_1 c_2 \cdots c_n : \forall i, 0 \leq c_i \leq n - i\}$. Define the *weight* of a code $c \in C_n$ to be $|c| = \sum_i c_i$. Then

$$\sum_{c \in C_n} q^{|c|} = (1)(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

- **Carlitz bijection:** Composite $\phi : S_n \to S_n$ of two weight-preserving bijections

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n.$$

Weight-preserving: $|\text{majcode}(\pi)| = \text{maj}(\pi)$ and $|\text{invcode}(\pi)| = \text{inv}(\pi)$.

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	

majcode(51423) =

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423		

$$\text{majcode}(51423) =$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	

$$\text{majcode}(51423) =$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2

$$\text{majcode}(51423) = 2$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123		

$$\text{majcode}(51423) = 2$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	

$$\text{majcode}(51423) = 2$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2

$$\text{majcode}(51423) = 22$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12		

$$\text{majcode}(51423) = 22$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	

$$\text{majcode}(51423) = 22$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	0

$$\text{majcode}(51423) = 220$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	0
1		

$$\text{majcode}(51423) = 220$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	0
1	0	

$$\text{majcode}(51423) = 220$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	0
1	0	0

$$\text{majcode}(51423) = 2200$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	0
1	0	0
$\emptyset$		

$$\text{majcode}(51423) = 2200$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	0
1	0	0
$\emptyset$	0	

$$\text{majcode}(51423) = 2200$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **majcode**: Remove entries starting with the largest, c_i records the amount maj decreases at the ith step:

Word	maj	c_i
51423	4	
1423	2	2
123	0	2
12	0	0
1	0	0
$\emptyset$	0	0

$$\text{majcode}(51423) = 22000$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions:

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions: $(3, 1)$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions: $(3, 1)$, $(3, 2)$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions: $(3, 1)$, $(3, 2)$, $(4, 1)$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions: $(3, 1)$, $(3, 2)$, $(4, 1)$, and $(4, 2)$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions: $(3, 1)$, $(3, 2)$, $(4, 1)$, and $(4, 2)$
- $c_1 = 2$, $c_2 = 2$, all other $c_i = 0$.

$$\text{invcode}(34125) = 22000$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions: $(3, 1)$, $(3, 2)$, $(4, 1)$, and $(4, 2)$
- $c_1 = 2$, $c_2 = 2$, all other $c_i = 0$.

$$\text{invcode}(34125) = 22000$$

- Since majcode(51423) $= 22000$ as well, the Carlitz bijection sends
$$51423 \to 34125.$$

The Carlitz bijection $S_n \to S_n$

$$S_n \xrightarrow{\text{majcode}} C_n \xleftarrow{\text{invcode}} S_n$$

- **invcode**: c_i is the number of inversions (j, i) where $i < j$.
- 34125 has four inversions: $(3, 1)$, $(3, 2)$, $(4, 1)$, and $(4, 2)$
- $c_1 = 2$, $c_2 = 2$, all other $c_i = 0$.

$$\text{invcode}(34125) = 22000$$

- Since $\text{majcode}(51423) = 22000$ as well, the Carlitz bijection sends
$$51423 \to 34125.$$
- Can extend the Carlitz bijection from permutations to words of any content.

Conjugate Symmetry in q and t

- Recall that
$$\widetilde{H}_\mu(X;q,t) = \widetilde{H}_{\mu^*}(X;t,q).$$
- Take the coefficient of $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots$ on both sides:

$$\sum_{\substack{\sigma:\mu\to\mathbb{Z}_+\\ |\sigma^{-1}(i)|=\alpha_i}} q^{\mathrm{inv}(\sigma)}t^{\mathrm{maj}(\sigma)} = \sum_{\substack{\rho:\mu^*\to\mathbb{Z}_+\\ |\rho^{-1}i|=\alpha_i}} q^{\mathrm{maj}(\rho)}t^{\mathrm{inv}(\rho)}.$$

Conjugate Symmetry in q and t

- Recall that
$$\widetilde{H}_\mu(X;q,t) = \widetilde{H}_{\mu^*}(X;t,q).$$
- Take the coefficient of $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ on both sides:
$$\sum_{\substack{\sigma:\mu\to\mathbb{Z}_+ \\ |\sigma^{-1}(i)|=\alpha_i}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} = \sum_{\substack{\rho:\mu^*\to\mathbb{Z}_+ \\ |\rho^{-1}i|=\alpha_i}} q^{\mathrm{maj}(\rho)} t^{\mathrm{inv}(\rho)}.$$
- Combinatorial proof: Need a bijection from fillings of μ to fillings of μ^* that preserves content and switches inv and maj simultaneously.

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c} \hline 6 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & \multicolumn{1}{c|}{7} \\ \hline 2 & 4 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c|}\hline 6 & 3 & \\ \hline 1 & 5 & 7 \\ \hline 2 & 4 & 1 \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\mathsf{maj}(\sigma) = \mathsf{maj}(612) + \mathsf{maj}(354) + \mathsf{maj}(71) = 1 + 2 + 1 = 4$

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c} \hline 6 & 3 & \multicolumn{1}{|c}{} \\ \hline 1 & 5 & \multicolumn{1}{|c|}{7} \\ \hline 2 & 4 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c} \hline 6 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & \multicolumn{1}{c|}{7} \\ \hline 2 & 4 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row:

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c|}\hline 6 & 3 & \\ \hline 1 & 5 & 7 \\ \hline \color{red}{2} & 4 & \color{red}{1} \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\mathrm{maj}(\sigma) = \mathrm{maj}(612) + \mathrm{maj}(354) + \mathrm{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row: $(2, 1)$

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c|}\cline{1-2} 6 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & 7 \\ \hline 2 & \color{red}{4} & \color{red}{1} \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row: $(2, 1)$, $(4, 1)$

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c|} \hline 6 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & 7 \\ \hline 2 & 4 & 1 \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row: $(2, 1)$, $(4, 1)$
- Second row: $(1, 5)$

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c} \hline 6 & 3 \\ \hline 1 & 5 & \multicolumn{1}{c|}{7} \\ \hline 2 & 4 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row: $(2, 1)$, $(4, 1)$
- Second row: $(1, 5)$, $(1, 7)$

inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c|} \hline 6 & 3 \\ \hline 1 & \color{red}{5} & \color{red}{7} \\ \hline 2 & \color{blue}{4} & 1 \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row: $(2, 1)$, $(4, 1)$
- Second row: $(1, 5)$, $(1, 7)$

inv and maj on fillings

$\sigma =$

6	3	
1	5	7
2	4	1

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row: $(2, 1)$, $(4, 1)$
- Second row: $(1, 5)$, $(1, 7)$
- Top row: $(6, 3)$

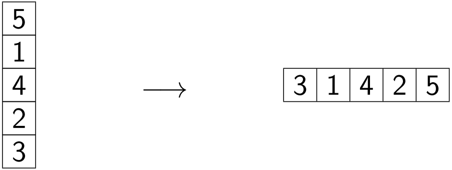
inv and maj on fillings

$$\sigma = \begin{array}{|c|c|c}\hline 6 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & \multicolumn{1}{c|}{7} \\ \hline 2 & 4 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

- maj is the sum of the maj's of the columns (top to bottom).
- $\text{maj}(\sigma) = \text{maj}(612) + \text{maj}(354) + \text{maj}(71) = 1 + 2 + 1 = 4$
- **Inversions:** A pair (a, b) with a to the left of b in a row is usually an inversion if and only if $a > b$, *except* if the entry c directly below a has value between a and b then the opposite is true.
- Bottom row: $(2, 1)$, $(4, 1)$
- Second row: $(1, 5)$, $(1, 7)$
- Top row: $(6, 3)$
- $\text{inv}(\sigma) = 5$

One-row shapes reduce to words

- If $\mu = (n)$ and σ is a filling of μ, then $\mathrm{maj}(\sigma) = 0$ and $\mathrm{inv}(\sigma) = \mathrm{inv}(w(\sigma))$ where $w(\sigma)$ is the reading word of σ.
- Similarly if ρ fills μ^* then $\mathrm{inv}(\rho) = 0$ and $\mathrm{maj}(\rho) = \mathrm{maj}(w(\sigma))$.



$\mathrm{maj} = 4$ $\qquad$ $\mathrm{inv} = 4$

$\mathrm{inv} = 0$ $\qquad$ $\mathrm{maj} = 0$

Result: Hook shapes (G.)

- Given a filling σ of a hook shape, define invcode and majcode according to the invcode and majcode of the row and column respectively.

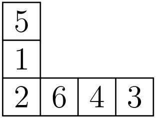
5			
1			
2	6	4	3

	1	2	3	4	5	6
invcode		0	2	1		0
majcode	0	0			1	

- Leftmost 0 of invcode matches rightmost 0 of majcode.

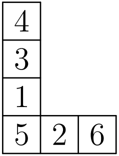
Result: Hook shapes (G.)

- Given a filling σ of a hook shape, define invcode and majcode according to the invcode and majcode of the row and column respectively.



	1	2	3	4	5	6
invcode		0	2	1		0
majcode	0	0			1	

- Leftmost 0 of invcode matches rightmost 0 of majcode.
- Now **interchange** and **reverse** the two codes!



	1	2	3	4	5	6
invcode		1			0	0
majcode	0		1	2	0	

Hall-Littlewood specialization: $q = 0$

- **Hall-Littlewood polynomials:** $\widetilde{H}_\mu(X;t) = \widetilde{H}_\mu(X;0,t)$

Hall-Littlewood specialization: $q = 0$

- **Hall-Littlewood polynomials:** $\widetilde{H}_\mu(X;t) = \widetilde{H}_\mu(X;0,t)$
- Symmetry problem restricts to fillings of μ with $\text{inv} = 0$ and fillings of μ^* with $\text{maj} = 0$. Need generalized Carlitz codes.

Hall-Littlewood specialization: $q = 0$

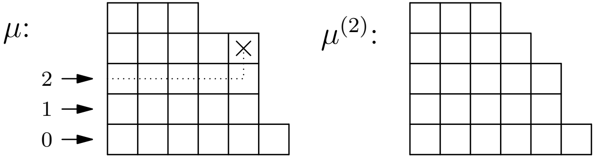
- **Hall-Littlewood polynomials:** $\widetilde{H}_\mu(X; t) = \widetilde{H}_\mu(X; 0, t)$
- Symmetry problem restricts to fillings of μ with $\mathrm{inv} = 0$ and fillings of μ^* with $\mathrm{maj} = 0$. Need generalized Carlitz codes.
- $\widetilde{H}_\mu(X; 0, t) = \mathrm{Frob}_t(R_\mu)$ for certain graded S_n-modules R_μ.

Hall-Littlewood specialization: $q = 0$

- **Hall-Littlewood polynomials:** $\widetilde{H}_\mu(X;t) = \widetilde{H}_\mu(X;0,t)$
- Symmetry problem restricts to fillings of μ with $\mathrm{inv} = 0$ and fillings of μ^* with $\mathrm{maj} = 0$. Need generalized Carlitz codes.
- $\widetilde{H}_\mu(X;0,t) = \mathrm{Frob}_t(R_\mu)$ for certain graded S_n-modules R_μ.
- (Garsia, Procesi.) $R_\mu = \mathbb{C}[z_1,\ldots,z_n]/I_\mu$ has a recursively defined monomial basis.

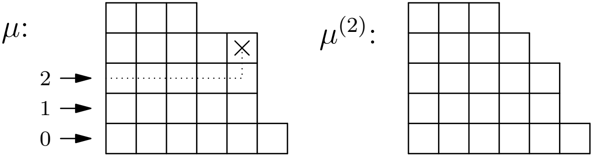
Hall-Littlewood specialization: $q = 0$

- **Hall-Littlewood polynomials:** $\widetilde{H}_\mu(X;t) = \widetilde{H}_\mu(X;0,t)$
- Symmetry problem restricts to fillings of μ with inv $= 0$ and fillings of μ^* with maj $= 0$. Need generalized Carlitz codes.
- $\widetilde{H}_\mu(X;0,t) = \mathrm{Frob}_t(R_\mu)$ for certain graded S_n-modules R_μ.
- (Garsia, Procesi.) $R_\mu = \mathbb{C}[z_1,\ldots,z_n]/I_\mu$ has a recursively defined monomial basis.
- Basis $\mathcal{B}_\mu$ given by recursion $\mathcal{B}_\mu = \bigsqcup_d z_n^d \cdot \mathcal{B}_{\mu^{(d)}}$ where $\mu^{(d)}$ is the shape formed by removing the corner in column μ_d.



Hall-Littlewood specialization: $q = 0$

- **Hall-Littlewood polynomials:** $\widetilde{H}_\mu(X;t) = \widetilde{H}_\mu(X;0,t)$
- Symmetry problem restricts to fillings of μ with $\mathrm{inv} = 0$ and fillings of μ^* with $\mathrm{maj} = 0$. Need generalized Carlitz codes.
- $\widetilde{H}_\mu(X;0,t) = \mathrm{Frob}_t(R_\mu)$ for certain graded S_n-modules R_μ.
- (Garsia, Procesi.) $R_\mu = \mathbb{C}[z_1,\dots,z_n]/I_\mu$ has a recursively defined monomial basis.
- Basis $\mathcal{B}_\mu$ given by recursion $\mathcal{B}_\mu = \bigsqcup_d z_n^d \cdot \mathcal{B}_{\mu^{(d)}}$ where $\mu^{(d)}$ is the shape formed by removing the corner in column μ_d.



- Define $C_\mu = \{c_1 \cdots c_n : z_n^{c_1} \cdots z_1^{c_n} \in \mathcal{B}_\mu\}$.

Generalizing Carlitz

Want weight-preserving bijections

$$\mathcal{F}_\mu|_{\mathsf{inv}=0} \xrightarrow{\mathsf{majcode}} C_\mu \xleftarrow{\mathsf{invcode}} \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0}$$

where

$$\mathcal{F}_\mu|_{\mathsf{inv}=0} = \{\sigma : \mu \to \mathbb{Z}_+ \mid \mathsf{inv}(\sigma) = 0\}$$

and

$$\mathcal{F}_{\mu^*}|_{\mathsf{maj}=0} = \{\rho : \mu^* \to \mathbb{Z}_+ \mid \mathsf{maj}(\rho) = 0\}.$$

Here the *weight* of a code $c \in C_\mu$ is $|c| = \sum c_i$, so the maj statistic on the left and the inv statistic on the right will be sent to this weight statistic on C_μ.

The map invcode

- Let ρ be a filling of μ^* having $\operatorname{maj}(\rho) = 0$. Order its entries by size with ties broken in reading order, forming a totally ordered alphabet $A = \{a_1, \ldots, a_n\}$.

$$\rho = \begin{array}{|c|c|c} \hline 3 & 2 \\ \hline 4 & 6 & \multicolumn{1}{c|}{1} \\ \hline 5 & 6 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$$A = 12234566$$

The map invcode

- Let ρ be a filling of μ^* having $\mathrm{maj}(\rho) = 0$. Order its entries by size with ties broken in reading order, forming a totally ordered alphabet $A = \{a_1, \ldots, a_n\}$.
- **Attacking pair**: (a, b) where $a > b$ and b is either to the right of a in the same row, or to the left of a in the row just below.

$\rho =$

3	2	
4	6	1
5	6	2

$A = 12234566$

The map invcode

- Let ρ be a filling of μ^* having $\text{maj}(\rho) = 0$. Order its entries by size with ties broken in reading order, forming a totally ordered alphabet $A = \{a_1, \ldots, a_n\}$.
- **Attacking pair**: (a, b) where $a > b$ and b is either to the right of a in the same row, or to the left of a in the row just below.
- Define $\text{invcode}(\rho) = c_1 c_2 \cdots c_n$ where c_i is the number of *attacking pairs* having a_i as its smaller entry.

$\rho =$

3	2	
4	6	1
5	6	2

$$A = 12234566$$

$$\text{invcode}(\rho) = 21200100$$

The map invcode

Theorem (G.)

The map invcode *is a weight-preserving bijection*

$$\mathrm{invcode} : \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0} \xrightarrow{\sim} C_\mu.$$

The map invcode

Theorem (G.)

The map invcode *is a weight-preserving bijection*

$$\mathsf{invcode} : \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0} \xrightarrow{\sim} C_\mu.$$

- We now have half of the desired bijection:

$$\mathcal{F}_\mu|_{\mathsf{inv}=0} \xrightarrow{\mathsf{majcode}} C_\mu \xleftarrow{\mathsf{invcode}} \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0}$$

The map invcode

Theorem (G.)

The map invcode *is a weight-preserving bijection*

$$\mathsf{invcode} : \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0} \xrightarrow{\sim} C_\mu.$$

- We now have half of the desired bijection:

$$\mathcal{F}_\mu|_{\mathsf{inv}=0} \xrightarrow{\mathsf{majcode}} C_\mu \xleftarrow{\mathsf{invcode}} \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0}$$

- To extend majcode to fillings, want to remove the largest entry and record how much the maj decreases by at each step.

The map invcode

Theorem (G.)

The map invcode *is a weight-preserving bijection*

$$\mathsf{invcode} : \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0} \xrightarrow{\sim} C_\mu.$$

- We now have half of the desired bijection:

$$\mathcal{F}_{\mu}|_{\mathsf{inv}=0} \xrightarrow{\mathsf{majcode}} C_\mu \xleftarrow{\mathsf{invcode}} \mathcal{F}_{\mu^*}|_{\mathsf{maj}=0}$$

- To extend majcode to fillings, want to remove the largest entry and record how much the maj decreases by at each step.
- How to remove largest entry from an inversion-free filling σ?

5		
7	1	2
3	4	6

A recursion and Killpatrick's map for distinct entries

- The recursion defining $\mathcal{B}_\mu$ implies that we should remove the largest entry from σ so that, if the major index decreases by d, the resulting shape is $\mu^{(d)}$.

A recursion and Killpatrick's map for distinct entries

- The recursion defining $\mathcal{B}_\mu$ implies that we should remove the largest entry from σ so that, if the major index decreases by d, the resulting shape is $\mu^{(d)}$.
- Killpatrick gave a combinatorial proof of this recursion in the language of the **charge** statistic on words. This translates to a map majcode for fillings σ having all distinct entries! ☺

A recursion and Killpatrick's map for distinct entries

- The recursion defining $\mathcal{B}_\mu$ implies that we should remove the largest entry from σ so that, if the major index decreases by d, the resulting shape is $\mu^{(d)}$.
- Killpatrick gave a combinatorial proof of this recursion in the language of the **charge** statistic on words. This translates to a map majcode for fillings σ having all distinct entries! ☺
- Unfortunately, does not extend to repeated entries in any way that preserves relative ordering of entries. ☹

A recursion and Killpatrick's map for distinct entries

- The recursion defining $\mathcal{B}_\mu$ implies that we should remove the largest entry from σ so that, if the major index decreases by d, the resulting shape is $\mu^{(d)}$.
- Killpatrick gave a combinatorial proof of this recursion in the language of the **charge** statistic on words. This translates to a map majcode for fillings σ having all distinct entries! ☺
- Unfortunately, does not extend to repeated entries in any way that preserves relative ordering of entries. ☹
- Via a different approach, can construct a map majcode for three-row shapes with general entries.

Results

Theorem (G.)

The Carlitz bijection can be extended to prove q, t-symmetry in the following cases:

- **Hook shapes:** *When μ is a hook shape.*

Results

Theorem (G.)

The Carlitz bijection can be extended to prove q, t-symmetry in the following cases:

- **Hook shapes:** *When μ is a hook shape.*
- $q = 0$**, distinct entries:** *For fillings with distinct entries (and any shape μ) when one of the statistics is zero.*

Results

Theorem (G.)

The Carlitz bijection can be extended to prove q, t-symmetry in the following cases:

- **Hook shapes:** *When μ is a hook shape.*
- $q = 0$**, distinct entries:** *For fillings with distinct entries (and any shape μ) when one of the statistics is zero.*
- $q = 0$**,** $\ell(\mu) \leq 3$**:** *When one of the statistics is zero and μ has at most three parts, showing that*

$$\widetilde{H}_{\mu}(X; 0, t) = \widetilde{H}_{\mu^*}(X; t, 0)$$

for such shapes.

Acknowledgments

- Thanks to Mark Haiman and Angela Hicks for their help and feedback.

Thank You!