# Number theoretic properties of generating functions related to Dyson's rank for partitions into distinct parts

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**Abstract**

Let $Q(n)$ denote the number of partitions of $n$ into distinct parts. We show that Dyson's rank provides a combinatorial interpretation of the well-known fact that $Q(n)$ is almost always divisible by 4. This interpretation gives rise to a new false theta function identity that reveals surprising analytic properties of one of Ramanujan's mock theta functions, which in turn gives generating functions for values of certain Dirichlet $L$-functions at non-positive integers.

## 1 Introduction and statement of results

A *partition* $\lambda$ of a positive integer $n$ is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers, written in nonincreasing order, whose sum is $n$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we say that $\lambda_i$ is the $i$th *part* of the partition, and we write $\ell(\lambda)$ to denote the number of parts of $\lambda$. The *rank* of $\lambda$ is $\lambda_1 - \ell(\lambda)$. For instance, the rank of $(5, 3, 1, 1)$ is $5 - 4 = 1$. The *Young diagram* of $\lambda$ is the partial grid of squares consisting of $k$ rows, aligned at the left, with the $i$th row containing $\lambda_i$ squares for each $i \leq k$. (See Figure 1.)

Let $p(n)$ denote the number of partitions of $n$. Ramanujan proved the following famous congruence identities for $p(n)$.

$$p(5n+4) \equiv 0 \pmod{5} \tag{1.1}$$
$$p(7n+5) \equiv 0 \pmod{7} \tag{1.2}$$
$$p(11n+6) \equiv 0 \pmod{11} \tag{1.3}$$

Several infinite families of arithmetic congruences for $p(n)$ have been discovered since Ramanujan's time, producing identities such as

$$p(157525693n + 111247) \equiv 0 \pmod{13}.$$

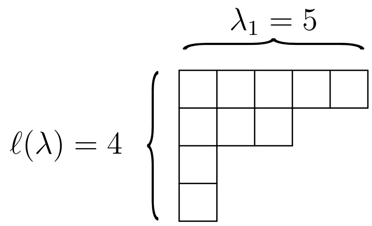


Figure 1: The partition $(5, 3, 1, 1)$. Notice that the rank of a partition is the difference between the width and the height of its Young diagram.

(See [16] for a detailed account of congruences for $p(n)$.)

Identities (1.1)-(1.3) simply begged for a combinatorial explanation. Dyson [11] conjectured that for any $m$, the number of partitions of $5n+4$ having rank congruent to $m \pmod 5$ is equal to $\frac{1}{5}p(5n+4)$, and the number of partitions of $7n+5$ having rank congruent to $m \pmod 7$ is equal to $\frac{1}{7}p(7n+5)$, thereby providing a combinatorial interpretation of (1.1) and (1.2) if true. Atkin and Swinnerton-Dyer proved these conjectures in [8]. Interestingly, equation (1.3) does not have a similar combinatorial interpretation given by the rank. Andrews and Garvan later discovered another combinatorial statistic, the crank, that classifies the partitions of $11n+6$ into 11 equal classes determined by the crank modulo 11. (See [6], [12].)

Let $Q(n)$ denote the number of partitions of $n$ into distinct parts. We call such partitions *strict* partitions. For instance, $(5, 3, 2)$ is a strict partition of 10, but $(5, 3, 1, 1)$ is not. Several infinite families of congruence identities have also been shown for $Q$. (See [14], [15], [18], [19].) In fact, it was shown in [15] that for any prime $p$, there exist positive integers $a$ and $b$ such that $Q(an+b) \equiv 0 \pmod p$ for all positive integers $n$.

The nearly arithmetic congruence identities modulo 4, first discovered by Rødseth [18], rival (1.1)-(1.3) in their simplicity. The first few such identities are:

$$\begin{aligned}
Q(5n+1) &\equiv 0 \pmod 4 && \text{if } n \not\equiv 0 \pmod 5 && (1.4)\\
Q(7n+2) &\equiv 0 \pmod 4 && \text{if } n \not\equiv 0 \pmod 7 && (1.5)\\
Q(11n+5) &\equiv 0 \pmod 4 && \text{if } n \not\equiv 0 \pmod{11} && (1.6)\\
Q(13n+7) &\equiv 0 \pmod 4 && \text{if } n \not\equiv 0 \pmod{13} && (1.7)
\end{aligned}$$

It turns out that $Q(n)$ is also highly divisible by arbitrary powers of 2. Gordon and Ono [13] have shown that for any positive integer $j$,

$$\lim_{N\to\infty} \frac{\#\{n < N \mid Q(n) \equiv 0 \pmod{2^j}\}}{N} = 1. \tag{1.8}$$

The proof of this fact depends on the theory of Galois representations and is not combinatorial. A simple combinatorial argument shows that $Q(n)$ is divisible by 2 if and only if $n \neq k(3k \pm 1)/2$ for any integer $k$, thus proving equation (1.8) in the case $j = 1$. Alladi [1] has provided combinatorial interpretations of (1.8) for $j \leq 4$.

| Rank (mod 4) | Partitions of 12 |
|---|---|
| 0 | $(10, 2)$, $(7, 4, 1)$, $(7, 3, 2)$ |
| 1 | $(11, 1)$, $(8, 3, 1)$, $(7, 5)$, $(5, 4, 2, 1)$ |
| 2 | $(9, 2, 1)$, $(8, 4)$, $(6, 3, 2, 1)$, $(5, 4, 3)$ |
| 3 | $(12)$, $(9, 3)$, $(6, 5, 1)$, $(6, 4, 2)$ |

| Rank (mod 4) | Partitions of 16 |
|---|---|
| 0 | $(14, 2)$, $(11, 4, 1)$, $(11, 3, 2)$, $(10, 6)$<br>$(8, 5, 2, 1)$, $(8, 4, 3, 1)$, $(7, 6, 3)$, $(7, 5, 4)$ |
| 1 | $(15, 1)$, $(12, 3, 1)$, $(11, 5)$, $(9, 4, 2, 1)$<br>$(8, 7, 1)$, $(8, 6, 2)$, $(8, 5, 3)$, $(6, 4, 3, 2, 1)$ |
| 2 | $(13, 2, 1)$, $(12, 4)$, $(10, 3, 2, 1)$, $(9, 6, 1)$<br>$(9, 5, 2)$, $(9, 4, 3)$, $(6, 5, 4, 1)$, $(6, 5, 3, 2)$ |
| 3 | $(16)$, $(13, 3)$, $(10, 5, 1)$, $(10, 4, 2)$<br>$(9, 7)$, $(7, 6, 2, 1)$, $(7, 5, 3, 1)$, $(7, 4, 3, 2)$ |

Table 1: The strict partitions of 12 and of 16 sorted by rank.

We show that Dyson's rank also provides a combinatorial interpretation of (1.4)-(1.7), and more generally of (1.8) for $j = 2$, as follows. Define $T(m, k; n)$ to be the number of strict partitions of $n$ having rank congruent to $m \pmod{k}$.

**Theorem 1.1.** *Let $n$ be a positive integer. We have*

$$T(0, 4; n) = T(1, 4; n) = T(2, 4; n) = T(3, 4; n)$$

*if and only if $24n + 1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$, and the largest power of $p$ dividing $24n + 1$ is $p^e$ where $e$ is odd.*

To illustrate Theorem 1.1, we sort the partitions of 12 and of 16 by rank in Table 1. Notice that $24 \cdot 12 + 1 = 289 = 17^2$, and so 12 does not satisfy the conditions of Theorem 1.1. On the other hand, $24 \cdot 16 + 1 = 385 = 5 \cdot 77$, so 16 satisfies the conditions with $p = 5$.

Notice that if $n$ satisfies the conditions of Theorem 1.1 then $Q(n) \equiv 0 \pmod{4}$. It is easily shown that this set of integers contains those of the form $pn + \frac{p^2-1}{24}$, $n \not\equiv 0 \pmod{p}$, for all primes $p > 3$ not congruent to $1 \pmod{24}$, thus proving (1.4)-(1.7) combinatorially via the rank.

Theorem 1.1 reveals fascinating properties of the generating functions related to the ranks of strict partitions. Let $Q(n, r)$ denote the number of partitions of $n$ into distinct parts with rank $r$, and define

$$G(z, q) = \sum_{n,r} Q(n, r) z^r q^n \tag{1.9}$$

where $n$ and $r$ range from 0 to $\infty$. The next theorem shows that the specializations of this series at fourth roots of unity $z$ have elegant and useful $q$-series expansions.

**Theorem 1.2.** *Let $z, q \in \mathbb{C}$ with $|z| \leq 1$, $|q| < 1$. Then*

$$G(z,q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(1-zq)(1-zq^2)\cdots(1-zq^s)},$$

*and we have*

$$\begin{aligned}
G(1,q) &= \sum_{n=0}^{\infty} Q(n)q^n = (1+q)(1+q^2)(1+q^3)\cdots \\
G(-1,q) &= \sum_{n=0}^{\infty} (T(n;0,2) - T(n;1,2))q^n \\
G(i,q) &= \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2} \\
G(-i,q) &= \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}
\end{aligned}$$

*where we define $Q(0) = 1$.*

The functions $G(1,q)$ and $G(-1,q)$ above have been studied rather extensively. Both of these functions are combinatorial generating functions that are related to automorphic forms, in the variable $\tau$ where $q = e^{2\pi i \tau}$ (we use this notation throughout). Since we have that

$$qG(1,q^{24}) = \frac{\eta(48\tau)}{\eta(24\tau)},$$

where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$ is the usual classical weight 1/2 modular form of Dedekind, it follows that $G(1,q)$ is essentially a weight 0 modular form. The series $G(-1,q)$ was thoroughly investigated by Andrews, Dyson, and Hickerson [4]. Their work shows that $G(-1,q)$ is related to the Fourier expansion of a Maass cusp form that has eigenvalue 1/4 with respect to the hyperbolic Laplacian operator $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, where $\tau = x + iy$. (See [10].)

This prompts one to ask if the functions the $G(z,q)$ have interesting analytic properties for all roots of unity $z$. Theorem 1.2 shows that these series have a simple form when $z = i$ and when $z = -i$. In fact, they are examples of *false theta functions*. To demonstrate this, we first recall some necessary background and notation. A *Dirichlet character* of order $a$ is a map $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

- $\chi(n + a) = \chi(n)$ for any integer $n$,
- $\chi(mn) = \chi(m)\chi(n)$ for any integers $m, n$, and
- $\chi(n) = 0$ for any $n$ such that $gcd(a, n) > 1$.

The eight Dirichlet characters of order 24 are shown in Table 2.

A *theta function* is a function $\theta(z;\tau)$, where $z$ is a fixed complex number and the domain of $\tau$ is the complex upper half plane $\mathcal{H}$, of the form

$$\theta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi inz} e^{2\pi in^2\tau} = \sum_{n=-\infty}^{\infty} e^{2\pi inz} q^{n^2}.$$

Several variants of these functions are also called theta functions if they satisfy certain modular transformation laws. In particular, if $\chi$ is an even Dirichlet character of order $a$, then

$$\sum_{n=-\infty}^{\infty} \chi(n) q^{n^2}$$

is a modular form of weight 1/2 over the congruence subgroup $\Gamma_0(4a^2)$ of the full modular group $PSL_2(\mathbb{Z})$. Moreover, these theta functions essentially form a basis of all modular forms of weight 1/2 by a classical theorem due to Serre and Stark [17].

Notice that, by Theorem 1.2,

$$qG(i, q^{24}) = \sum_{k=0}^{\infty} i^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} i^{k-1} q^{(6k-1)^2}.$$

This closely resembles the theta functions described above, but the coefficients cannot be written as a linear combination of even Dirichlet characters. Thus, we have encountered a false theta function.

False theta function identities can be used to obtain generating functions for the values of Dirichlet $L$-functions at non-positive integers. This was first observed by Andrews, Ono and Urroz [5], and by Zagier [20]. Here we show that the identities in Theorem 1.2 also may be used in this way. We first recall the definition of $L$-functions. Given a Dirichlet character $\chi$, the corresponding Dirichlet $L$-function is a generalization of the Riemann $\zeta$-function defined by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.10}$$

| $n$ | 1 | 5 | 7 | 11 | 13 | 17 | 19 | 23 |
|---|---|---|---|---|---|---|---|---|
| $\chi_0(n)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_1(n)$ | 1 | 1 | $-1$ | $-1$ | 1 | 1 | $-1$ | $-1$ |
| $\chi_2(n)$ | 1 | $-1$ | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 |
| $\chi_3(n)$ | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\chi_4(n)$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| $\chi_5(n)$ | 1 | $-1$ | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_6(n)$ | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| $\chi_7(n)$ | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ | 1 | 1 |

Table 2: The nonzero values of the 8 Dirichlet characters of order 24.

Each $L$-function has an analytic continuation to the entire complex plane. In Section 2.3, we use our expressions for $G(\pm i, z)$ to obtain the following.

**Theorem 1.3.** *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_6, -2n) t^n = 2e^{-t} + \sum_{n=1}^{\infty} \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1-ie^{-24rt})} + \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1+ie^{-24rt})} \tag{1.11}$$

$$= e^{-t} + e^{-t} \sum_{n=1}^{\infty} \frac{e^{-24nt}}{\prod_{r=1}^{n}(1+e^{-48rt})} \tag{1.12}$$

*and*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_7, -2n) t^n = i \sum_{n=1}^{\infty} \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1-ie^{-24rt})} - \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1+ie^{-24rt})} \tag{1.13}$$

The $L$-values at negative integers can also be obtained using generalized Bernoulli numbers. The Bernoulli numbers $B_{n,\chi}$ associated with the Dirichlet character $\chi$ of order $a$ are defined by the generating function

$$\sum_{m=1}^{a} \chi(m) \frac{te^{mt}}{e^{at}-1} = \sum_{t=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

It is well-known that

$$L(\chi, 1-k) = -\frac{B_{k,\chi}}{k}$$

for any positive integer $k$. The right hand side of (1.11) is, as a power series in $t$,

$$2 + 46t + 3985t^2 + \frac{1743623}{3}t^3 + \cdots$$

which matches the values given by the Bernoulli numbers for $\chi_6$. As another illustration, the right hand side of (1.13) is

$$-48t - 3984t^2 - 581208t^3 - \cdots.$$

In addition to being a false theta function, $G(i, q)$ is related to the famous mock theta functions of Ramanujan, which Bringmann and Ono [9] recently have established are the holomorphic parts of certain weight 1/2 harmonic Maass forms. One famous such function is

$$R(z, q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-zq^k)(1-z^{-1}q^k)}. \tag{1.14}$$

The coefficient of $z^m q^n$ in $R(z, q)$ is the number of partitions of $n$ having rank $m$. Thus, evaluating $R(z, q)$ at roots of unity $z$ is useful in obtaining congruence relations for $p(n)$ via the rank.

Replacing $q$ by $1/q$ in (1.14), we obtain the following theorem.

**Theorem 1.4.** *We have*

$$R(i,1/q) = R(-i,1/q) = \frac{1-i}{2}G(i,q) + \frac{1+i}{2}G(-i,q)$$

*or alternatively,*

$$\begin{aligned} qR(i,1/q^{24}) &= \sum_{n=0}^{\infty}(-1)^n \left(q^{(12n+1)^2} + q^{(12n+5)^2} + q^{(12n+7)^2} + q^{(12n+11)^2}\right) \\ &= q + q^{25} + q^{49} + q^{121} - q^{169} - q^{289} - q^{361} - q^{529} + q^{625} + \cdots . \end{aligned}$$

Thus, the analytic behavior of the false theta functions $G(\pm i, q)$ gives the behavior of $R(\pm i; q)$ for $q$ outside the unit disk! This is a remarkable connection between the rank generating functions of strict and unrestricted partitions.

# 2 Proofs

We now present the proofs of the main results.

## 2.1 $Q(n)$ mod 4 via the rank

Let $\mathcal{D}$ denote the set of all strict partitions (partitions having distinct parts), and let $\mathcal{P}$ denote the set of all (unrestricted) partitions. Let $\mathcal{D}_n$ denote the set of all strict partitions of $n$.

Define a *pentagonal partition* to be a partition of the form $(2k, 2k-1, \ldots, k+1)$ or $(2k-1, 2k-2, \ldots, k)$ for some positive integer $k$. The former is a partition of $k(3k+1)/2$, and the latter is a partition of $k(3k-1)/2$. Numbers of the form $k(3k\pm 1)/2$ are called *pentagonal numbers*. An example of each type of pentagonal partition is shown in Figure 2.

Let $\mathcal{D'}_n$ denote the set of all strict partitions of $n$ that are not pentagonal partitions. For any partition $\lambda$, let $m(\lambda)$ be the largest index $m$ such that $\lambda_1 = \lambda_2 + 1 = \lambda_3 + 2 = \cdots = \lambda_m + m - 1$. Also let $s(\lambda)$ denote the smallest part of $\lambda$.

Given a partition $\lambda$, the *conjugate partition* of $\lambda$, denoted $\lambda'$, is the partition formed by interchanging the rows and columns of its Young diagram.

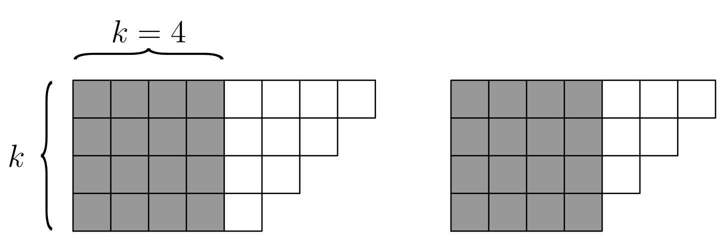


Figure 2: The two types of pentagonal partitions of length $k$, shown here for $k = 4$.

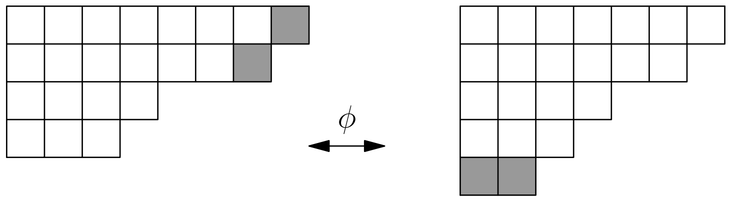

Figure 3: Franklin's Involution $\phi : \mathcal{D'}_n \to \mathcal{D'}_n$.

To prove Theorem 1.1, we first provide a necessary and sufficient condition for the equalities $T(0,4;n) = T(2,4;n)$ and $T(1,4;n) = T(3,4;n)$ to hold.

**Lemma 2.1.** *If $n \neq k(3k \pm 1)/2$ for any $k$, we have*

$$T(1,4;n) = T(3,4;n) \text{ and } T(0,4;n) = T(2,4;n).$$

*Otherwise, if $n = k(3k+1)/2$, then*

$$T(k,4;n) = T(k+2,4;n) + 1 \text{ and } T(k+1,4;n) = T(k+3,4;n).$$

*If $n = k(3k-1)/2$, then*

$$T(k-1,4;n) = T(k+1,4;n) + 1 \text{ and } T(k,4;n) = T(k+2,4;n).$$

*Proof.* We require an involution $\phi : \mathcal{D'}_n \to \mathcal{D'}_n$, commonly known as *Franklin's Involution*, defined as follows. Let $\lambda \in \mathcal{D'}_n$, and let $m = m(\lambda)$ and $s = s(\lambda)$. If $s \leq m$, define $\phi(\lambda)$ to be the partition formed by removing the part $s$ from the partition and increasing each of the first $s$ parts by 1. If $s > m$, define $\phi(\lambda)$ to be the partition formed by decreasing each of the first $m$ parts of $\lambda$ by 1 and inserting a part of size $m$ into the partition. (See Figure 3.) Notice that these operations are not well defined on pentagonal partitions.

It is easily verified that $\phi$ is an involution. Furthermore, for any non-pentagonal partition $\lambda$, the rank of $\phi(\lambda)$ differs from the rank of $\lambda$ by $\pm 2$. Thus, if $n \neq k(3k \pm 1)/2$, we have $T(1,4;n) = T(3,4;n)$ and $T(0,4;n) = T(2,4;n)$.

If $n = k(3k+1)/2$, there is one pentagonal partition of $n$, namely $(2k, 2k-1, \ldots, k+1)$, and the rank of this partition is $k$. Thus $T(k,4;n) = T(k+2,4;n)+1$ and $T(k+1,4;n) = T(k+3,4;n)$.

If $n = k(3k-1)/2$, there is one pentagonal partition of $n$, namely $(2k-1, 2k-2, \ldots, k)$, and the rank of this partition is $k-1$. Thus $T(k-1,4;n) = T(k+1,4;n)+1$ and $T(k,4;n) = T(k+2,4;n)$. This completes the proof. $\square$

Now, notice that $T(0,4;n) + T(2,4;n) = T(0,2;n)$ and $T(1,4;n) + T(3,4;n) = T(1,2;n)$. Thus, by Lemma 2.1, in order to find exactly when $T(m,4;n) = \frac{1}{4}Q(n)$ for $m = 0,1,2,3$ it suffices to find the difference between the number of partitions of $n$ having even rank, $T(0,2;n)$, and the number having odd rank, $T(1,2;n)$. Let $S(n) = T(0,2;n) - T(1,2;n)$ be this difference. An explicit formula for the function $S(n)$ has already been obtained [4], and we state this result below.

**Theorem 2.2.** *We have $S(n) = T(24n+1)$, where the function $T(m)$ is defined on the set of integers $m \neq 1$ congruent to $1 \pmod 6$ as follows. Write $m$ in the form $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where each $p_i$ is either a prime congruent to $1 \pmod 6$ or the negative of a prime congruent to $5 \pmod 6$. Then $T(m) = T(p_1^{e_1})T(p_2^{e_2}) \cdots T(p_k^{e_k})$, where*

$$T(p^e) = \begin{cases} 0 & \text{if } p \not\equiv 1 \pmod{24} \text{ and } e \text{ is odd} \\ 1 & \text{if } p \equiv 13 \text{ or } 19 \pmod{24} \text{ and } e \text{ is even} \\ (-1)^{e/2} & \text{if } p \equiv 1 \pmod{24} \text{ and } e \text{ is even} \\ e+1 & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = 2 \\ (-1)^e(e+1) & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = -2. \end{cases}$$

It follows that $S(n) = 0$ if and only if $24n+1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$, and the largest power of $p$ dividing $24n+1$ is $p^e$ for some odd positive integer $e$. Suppose $n$ is a pentagonal number. Then $24n+1 = 24(k(3k \pm 1)/2) + 1 = (6k \pm 1)^2$ for some $k$, so $n$ does not satisfy the conditions of Theorem 2.2. Thus, if $S(n) = 0$ then $n$ is not a pentagonal number, and so by Lemma 2.1, $T(0,4;n) = T(2,4;n)$ and $T(1,4;n) = T(3,4;n)$. Furthermore, if $S(n) = 0$ then $T(0,4;n) + T(2,4;n) = T(0,2;n) = T(1,2;n) = T(1,4;n) + T(3,4;n)$ by the definition of $S(n)$. Thus $S(n) = 0$ if and only if $T(0,4;n) = T(1,4;n) = T(2,4;n) = T(3,4;n)$. This proves Theorem 1.1.

To analyze the generating functions that arise in studying $S(n)$ and other functions related to the rank, we first recall some standard notation. For any positive integer $n$, we define

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The *Sylvester's triangle* of a partition $\lambda$ is the largest partition of the form $(s, s-1, \ldots, 3, 2, 1)$ such that $s - i + 1 \leq \lambda_i$ for $i = 1, 2, \ldots, s$. An example is shown in Figure 4. Notice that if $\lambda$ is a strict partition, $\lambda$ has the same number of parts as its Sylvester's triangle.

We proceed to prove Theorem 1.2, which we restate below. Recall that

$$G(z,q) = \sum_{n,r} Q(n,r) q^n z^r$$

where $Q(n,r)$ denotes the number of strict partitions of $n$ having rank $r$.

**Theorem 2.3.** *Let $z, q \in \mathbb{C}$ with $|q| < 1$. Then*

$$G(z,q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(zq;q)_s}$$

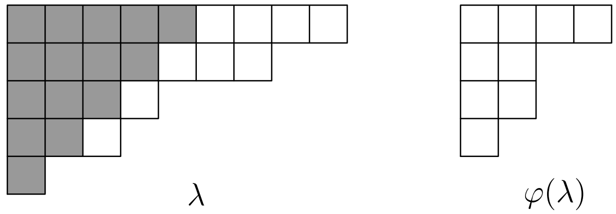


Figure 4: A partition $\lambda$ with its Sylvester's triangle shaded and its image under $\varphi$.

*and*

$$\begin{aligned}
G(1,q) &= \sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty \\
G(-1,q) &= \sum_{n=0}^{\infty} S(n)q^n \\
G(i,q) &= \sum_{k=1}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2} \\
G(-i,q) &= \sum_{k=1}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}.
\end{aligned}$$

*Proof.* Let $Q(n,r,s)$ denote the number of strict partitions with rank $r$ and exactly $s$ parts, and let $p(n,r,s)$ denote the number of partitions of $n$ with largest part at most $s$ and exactly $r$ parts. It is easily verified combinatorially that for any positive integer $s$,

$$\sum_{n,r} p(n,r,s)q^n z^r = \frac{1}{(zq;q)_s} \tag{2.1}$$

We now define a map $\varphi : \mathcal{D} \to \mathcal{P}$ as follows. Suppose $\lambda$ is a partition of $n$ into $s$ distinct parts.

By removing the Sylvester's triangle from $\lambda$, we are left with a partition $\nu = (\lambda_1 - s, \lambda_2 - (s-1), \ldots, \lambda_s - 1)$ of $n - s(s+1)/2$. We define $\varphi(\lambda)$ to be the conjugate partition $\nu'$ of $\nu$. Notice that $\nu'$ has at most $s$ parts, and the number of parts of $\nu'$ is equal to the rank of $\lambda$.

For each nonnegative integer $s$, $\varphi$ is a bijection from the set of partitions of $n$ into exactly $s$ distinct parts to the set of partitions $\nu'$ having largest part at most $s$. Hence

$$\sum_{n,r,s} Q(n,r,s)q^n z^r = \sum_s q^{s(s+1)/2} \sum_{n,r} p(n,r,s)q^n z^r \tag{2.2}$$

where the variables range over all nonnegative integers. Note that

$$\sum_s Q(n,r,s) = Q(n,r). \tag{2.3}$$

By (2.1), (2.2), and (2.3), we have

$$\sum_{n,r} Q(n,r) q^n z^r = 1 + \sum_{s=1}^{\infty} q^{s(s+1)/2} \frac{1}{(zq;q)_s} = G(z,q).$$

Setting $z = 1$, we have

$$G(1,q) = \sum_{n=0}^{\infty} Q(n) q^n,$$

and it is well-known that this summation is also equal to $(-q;q)_\infty$. (See [3], p. 5.)

Setting $z = -1$, we have

$$\begin{aligned} G(i,q) &= \sum_{n,r} Q(n,r)(-1)^r q^n \\ &= \sum_{n} (T(0,2;n) - T(1,2;n)) q^n \\ &= \sum_{n=0}^{\infty} S(n) q^n. \end{aligned}$$

Setting $z = i$, we have

$$\begin{aligned} G(i,q) &= \sum_{n,r} Q(n,r) i^r q^n \\ &= \sum_{n} [T(0,4;n) + iT(1,4;n) - T(2,4;n) - iT(3,4;n)] q^n \\ &= \sum_{n} [(T(0,4);n) - T(2,4;n)) + i(T(1,4;n) - T(3,4;n))] q^n \\ &= \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2} \end{aligned}$$

where the last equality follows from Lemma 2.1. Analogously,

$$G(-i,q) = \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}.$$

This completes the proof. $\square$

## 2.2 The relation between $R(\pm i, q)$ and $G(\pm i, q)$

To prove Theorem 1.4, we require the following identity given in Ramanujan's "lost" notebook:

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(-aq;q)_n(-a^{-1}q;q)_n}=(1+a)\sum_{n=0}^{\infty}a^{3n}q^{\frac{n(3n+1)}{2}}(1-a^2q^{2n+1})-\frac{a\sum_{n=0}^{\infty}(-1)^n a^{2n}q^{\frac{n(n+1)}{2}}}{(-aq;q)_\infty(-a^{-1}q;q)_\infty}. \tag{2.4}$$

Andrews [2] noted that by substituting $a=i$ in (2.4) and taking the real part of both sides, we obtain the identity

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(-q^2;q^2)_n}=\sum_{n=0}^{\infty}(-1)^nq^{n(6n+1)}(1+q^{4n+1})+\sum_{n=0}^{\infty}(-1)^nq^{(2n+1)(3n+2)}(1+q^{4n+3}). \tag{2.5}$$

Notice that the left hand side of (2.5) is equal to $R(i,1/q)$. Replacing $q$ by $q^{24}$ in 2.5 and multiplying by $q$, we obtain, by Theorem 1.2,

$$\begin{aligned} qR(i,1/q^{24}) &= \sum_{n=0}^{\infty}(-1)^n\left(q^{(12n+1)^2}+q^{(12n+5)^2}+q^{(12n+7)^2}+q^{(12n+11)^2}\right)\\ &= \frac{1-i}{2}qG(i,q^{24})+\frac{1+i}{2}qG(-i,q^{24}) \end{aligned}$$

and the result follows.

## 2.3 Exponential generating functions for Dirichlet L-values

In order to prove Theorem 1.3, we first prove the following.

**Lemma 2.4.** *Let $\chi_6$ and $\chi_7$ denote the Dirichlet characters of order 24 given in Table 2, and let $0\le t<1$. We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{1+i}{2}L(\chi_6,-2n)+\frac{1-i}{2}L(\chi_7,-2n)\right)t^n=e^{-t}+\sum_{n=1}^{\infty}\frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1-ie^{-24rt})} \tag{2.6}$$

*and*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{1-i}{2}L(\chi_6,-2n)+\frac{1+i}{2}L(\chi_7,-2n)\right)t^n=e^{-t}+\sum_{n=1}^{\infty}\frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1+ie^{-24rt})}. \tag{2.7}$$

*Proof.* Define $H:\mathbb{R}\to\mathbb{C}$ by

$$H(t)=e^{-t}+\sum_{n=1}^{\infty}\frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1-ie^{-24rt})}.$$

By Theorem 1.2, for $t>0$,

$$H(t)=e^{-t}G(i,e^{-24t})=\sum_{k=0}^{\infty}i^ke^{-(6k+1)^2t}+\sum_{k=1}^{\infty}i^{k-1}e^{-(6k-1)^2t}. \tag{2.8}$$

Notice that $\frac{(1+i)}{2}\chi_6(6k+1)+\frac{(1-i)}{2}\chi_7(6k+1) = i^k$ and $\frac{(1+i)}{2}\chi_6(6k-1)+\frac{(1-i)}{2}\chi_7(6k-1) = i^{k-1}$, and that $\chi_6(n) = \chi_7(n) = 0$ when $n$ is not of the form $6k+1$ or $6k-1$. Thus (2.8) becomes

$$H(t) = \sum_{n=0}^{\infty} \left( \frac{(1+i)}{2}\chi_6(n) + \frac{(1-i)}{2}\chi_7(n) \right) e^{-n^2 t}.$$

Now, let $F : \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} \to \mathbb{C}$ by $F(s) = \int_0^\infty H(t)t^{s-1}dt$. For any $s$ with $\mathrm{Re}(s) > 0$, we have

$$\begin{aligned} F(s) &= \int_0^\infty \sum_{n=0}^{\infty} \left( \frac{(1+i)}{2}\chi_6(n) + \frac{(1-i)}{2}\chi_7(n) \right) e^{-n^2 t} t^{s-1} dt && (2.9) \\ &= \sum_{n=0}^{\infty} \left( \frac{(1+i)}{2}\chi_6(n) + \frac{(1-i)}{2}\chi_7(n) \right) \int_0^\infty e^{-n^2 t} t^{s-1} dt && (2.10) \end{aligned}$$

since the integral and sum are absolutely convergent for $\mathrm{Re}(s) > 0$. Recall that the $\Gamma$ function is commonly defined as $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$. Substituting $u = n^2 t$ in the integral in each summand in (2.10), we find

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} \left( \frac{(1+i)}{2}\chi_6(n) + \frac{(1-i)}{2}\chi_7(n) \right) \frac{1}{n^{2s}} \int_0^\infty e^{-u} u^{s-1} dt \\ &= \Gamma(s) \left( \frac{(1+i)}{2} \sum_{n=0}^{\infty} \frac{\chi_6(n)}{n^{2s}} + \frac{(1-i)}{2} \sum_{n=0}^{\infty} \frac{\chi_7(n)}{n^{2s}} \right) \\ &= \Gamma(s) \left( \frac{(1+i)}{2} L(\chi_6, 2s) + \frac{(1-i)}{2} L(\chi_7, 2s) \right) \end{aligned}$$

It is well-known ([7], p. 250) that $\Gamma$ has an analytic continuation to $\mathbb{C} \backslash \{n \in \mathbb{Z} : n \leq 0\}$, with poles of order 1 at the nonpositive integers, defined as follows. For any negative integer $n$ and any $s$ with $n < \mathrm{Re}(s) \leq n+1$, $\Gamma(s) = \frac{1}{s(s+1)\cdots(s+n-1)}\Gamma(s+n)$. It is easily verified that the residue of $\Gamma$ at the negative integer $k$ is $(-1)^n/n!$.

Using the analytic continuations of $L(\chi_6, s)$ and $L(\chi_7, s)$, we can extend $F(s)$ to a meromorphic function on $\mathbb{C}$ that has poles of order 1 at the nonpositive integers and is analytic elsewhere. Moreover, the residue at the pole $s = -n$ of $F(s)$ is

$$\frac{(-1)^n}{n!} \left( \frac{(1+i)}{2} L(\chi_6, -2n) + \frac{(1-i)}{2} L(\chi_7, -2n) \right).$$

Define the complex numbers $a(n)$ by

$$H(t) = \sum_{n=0}^{\infty} a(n) t^n,$$

since $H$ is analytic. Then, for any positive integer $N$,

$$\begin{aligned}\int_0^\infty H(t)t^{s-1}dt &= \int_0^1 \sum_{n=0}^\infty a(n)t^{n+s-1}dt + \int_1^\infty H(t)t^{s-1}dt \\ &= \sum_{n=0}^N \frac{a(n)}{n+s} + \sum_{n=N+1}^\infty \frac{a(n)}{n+s} + \int_1^\infty H(t)t^{s-1}dt.\end{aligned}$$

Since $\sum_{n=N+1}^\infty \frac{a(n)}{n+s} + \int_1^\infty H(t)t^{s-1}dt$ is an analytic function of $s$ in the half plane $\mathrm{Re}(s) > N$, the residue of the pole at $s = -n$ is $a(n)$. Thus

$$a(n) = \frac{(-1)^n}{n!}\left(\frac{(1+i)}{2}L(\chi_6, -2n) + \frac{(1-i)}{2}L(\chi_7, -2n)\right)$$

for all $n$, and equation (2.6) follows.

The proof of equation (2.7) is analogous. $\square$

Adding and subtracting equations (2.6) and (2.7), we obtain

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!}L(\chi_6, -2n)t^n = 2e^{-t} + \sum_{n=1}^\infty \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^n(1-ie^{-24rt})} + \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^n(1+ie^{-24rt})}$$

and

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!}L(\chi_7, -2n)t^n = i\sum_{n=1}^\infty \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^n(1-ie^{-24rt})} - \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^n(1+ie^{-24rt})},$$

which are equations (1.11) and (1.13) of Theorem 1.3.

To prove equality (1.12) of Theorem 1.3, note that by Theorem 1.4,

$$qR(i, 1/q^{24}) = \sum_{n=0}^\infty \chi_6(n)q^{n^2}.$$

Replacing $q$ by $e^{-t}$, an argument identical to that for Lemma 2.4 shows that

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!}L(\chi_6, -2n)t^n = e^{-t} + e^{-t}\sum_{n=1}^\infty \frac{e^{-24nt}}{\prod_{r=1}^n(1+e^{-48rt})}$$

as desired.

# 3 Future Work

Given the fascinating properties of the functions $G(z,q)$ and $R(z,q)$ when $z$ is a fourth root of unity, it is natural to ask whether Theorems 1.2 and 1.4 are specializations of a more general phenomenon that occurs when $z$ is an arbitrary root of unity. Understanding the behavior of the coefficients of these functions at $m$th roots of unity may also unlock more information about the distribution of the rank function modulo $m$, for both strict and unrestricted partitions.

Dyson's rank does not provide a combinatorial interpretation of the fact that $Q(n)$ is usually divisible by 8 in the same manner as it does for 2 and 4. Thus, it may also be fruitful to investigate generalizations of Dyson's rank in order to find a partition statistic that, when taken modulo $2^j$, classifies the partitions of $n$ into $2^j$ equal classes. More generally, perhaps an analog of the crank function that applies to $Q(n)$ is waiting to be discovered.

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