Math 672: Projective Geometry I
Homework 1

Hand in between 2 and 5 inclusive of the problems below; mark at the top of your page which problem numbers you are handing in. Write full proofs for each of your solutions. Your score will be as a percentage of those you hand in. To submit your solutions, either type on Overleaf/LaTeX or handwrite and take pictures for file uploads, and upload them to the Canvas course under the appropriate Assignment.

Problems

1. Given an example of an open set containing the point $(0:1:0)$ in $\mathbb{P}^2_{\mathbb{R}}$ (that is not the whole space).
2. Show that $\mathbb{P}^1_{\mathbb{C}}$ is topologically a sphere, and $\mathbb{P}^1_{\mathbb{R}}$ is topologically a circle.
3. Describe a projective transformation that takes the circle $x^2 + y^2 = z^2$ to the hyperbola $xy = z^2$.
4. Prove that an invertible projective transformation of $\mathbb{P}^1$, over any field of characteristic 0, is uniquely determined by where it sends three distinct points. (Recall that such a transformation is of the form $(x : y) \mapsto (ax + by : cx + dy)$, and we can consider the three points $(x : y)$ to be $(0 : 1)$, $(1 : 0)$, $(1 : 1)$.)
5. Show that the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$ intersect in two points in ordinary Euclidean real space $\mathbb{R}^2$, but in projective complex space $\mathbb{P}^2_{\mathbb{C}}$, their homogenizations $x^2 + y^2 = z^2$ and $(x-1)^2 + y^2 = z^2$ have four complex solutions in homogeneous coordinates, matching the expected output from Bezout's theorem.
6. Give an example showing that Bezout's theorem does not necessarily hold if we relax the condition that X is nonsingular.
7. Shafarevich section 1 problem 2: Which rational functions $p(x)/q(x)$ are regular at the point at infinity of $\mathbb{P}^1$? What order of zero do they have there?
8. Shafarevich section 1 problem 6(a): Prove that the sum of multiplicities of two singular points of an irreducible curve of degree n is at most n.
9. Shafarevich section 1 problem 6(b): and the sum of multiplicities of any 5 points is at most $2n$.
10. Shafarevich section 1 problem 12: Give an interpretation of Pascal's theorem in the case that pairs of vertices of the hexagon coincide, and the lines joining them become tangents.