Springer Theory:

Springer correspondence:

$$\begin{pmatrix} \text{Irred. reps} \\ \text{of Weyl group} \\ W_R \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Nilpotent conjugacy} \\ \text{classes in } \mathfrak{g} \\ (+\text{ a rep of an} \\ \text{associated gp } A_G) \end{pmatrix}$$

explicit bijection
via the Springer resolution,
coming from geometry of
flag varieties.

Flag varieties and Lie types

Def: Flag:

$$V_\bullet = \left(0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n\right)$$

$$\dim V_i = i$$

Ex: $0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle \subseteq \cdots \subseteq \mathbb{C}^n$

$e_1, \dots, e_n$ basis

GL_n action on flags: $g \in GL_n$

$$gV_\bullet = \left(0 = gV_0 \subseteq gV_1 \subseteq gV_2 \subseteq \cdots \subseteq gV_n = \mathbb{C}^n\right)$$

Ex: $E_\bullet = 0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \mathbb{C}^3$

$g = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \rightsquigarrow 0 \subseteq \langle e_2 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \mathbb{C}^3 = gE_\bullet$

<u>Lemma:</u> GL_n action is <u>transitive</u>: for any two flags $V_\bullet$, $W_\bullet$, $\exists g \in GL_n$,

$$gV_\bullet = W_\bullet$$

<u>Proof:</u> $g_v = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \Rightarrow gE_\bullet = V_\bullet$

$$g_w = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix}$$

$$g_w g_v^{-1} V_\bullet = g_w E_\bullet = W_\bullet$$

□

<u>Stabilizer of $E_\bullet$:</u>

$ge_1 = ce_1$ for some $c \neq 0$

$$\Rightarrow g = \begin{pmatrix} c & \\ 0 & \\ 0 & \cdots \\ \vdots & \\ 0 & \end{pmatrix}$$

$g\langle e_1, e_2\rangle = \langle ce_1, ge_2\rangle$ fixed iff $ge_2 = ae_1 + be_2$, $b \neq 0$

$$g = \begin{pmatrix} c & a & \\ & b & \cdots \\ & & \ddots \end{pmatrix}$$

...

g upper triangular!

$\Rightarrow \operatorname{stab}(E_\bullet) = B$, Borel subgroup

$\Rightarrow GL_n/B \cong \{\text{flags in } \mathbb{C}^n\}$

$\uparrow$ "flag variety"

→ Has topological structure inherited from Lie gp GL_n

→ Algebraic variety as well; "Plücker relations"

<u>Type A flag variety</u>: $GL_n/B \cong SL_n/B'$

$B' =$ Borel in SL_n, $\det = 1$

<u>Type B flag variety</u>: SO_{2n+1}/B

$\uparrow$ its <u>Borel</u> - maximal closed connected solvable subgp

Solvable: $\exists$ chain of normal subgps w/ abelian successive quotients:

$1 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B$

Alternatively: successive commutator subgps eventually reach the trivial gp.

In type A: $[B,B] = \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix} = B'$

$[B',B'] = \begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

etc.

<u>Type C</u>: SP_{2n}/B

<u>Type D</u>: SO_{2n}/B.

Type A: cohomology and S_n action

$H^*(Fl_n) =$ graded ring whose
- elements $\leftrightarrow$ formal lin. combinations of closed subvarieties (graded by codim) up to equiv.
- Product $\leftrightarrow$ intersection
- sum $\leftrightarrow$ union

(Equivalent to "chow ring", from 502)

Thm: $H^*(Fl_n) = \mathbb{C}[x_1, \dots, x_n]/(e_1, \dots, e_n)$ "coinvariant ring"

Ex: $n=1$ $\mathbb{C}[x_1]/e_1 = \mathbb{C}[x_1]/x_1 = \mathbb{C}$. $Fl_1 =$ one flag ✓

Ex: $n=2$ $\mathbb{C}[x_1, x_2]/(e_1, e_2) = \mathbb{C}[x_1, x_2]/(x_1 + x_2, x_1 x_2)$

$= \mathbb{C}[x_1]/(-x_1^2)$

$= \mathbb{C} \oplus \mathbb{C}\langle x_1 \rangle$

Ex: $n=3$ $\mathbb{C}[x_1, x_2, x_3]/(x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3)$

- Replace $x_1 = -x_3 - x_2$:

$$= \mathbb{C}[x_2,x_3] \big/ \left(x_2x_3 - (x_2+x_3)^2,\ x_2x_3^2 + x_3x_2^2\right)$$

$$= \mathbb{C}[x_2,x_3] \big/ \left(x_2^2 + x_2x_3 + x_3^2,\ x_2x_3(x_2+x_3)\right)$$

Basis: $1,\ x_2,\ x_3,\ x_2x_3,\ x_3^2,\ x_2x_3^2$ $\}$ 6-dim'l

(can express x_2^2 in terms of others,
$x_3^3 = \underbrace{x_3(x_2^2 + x_2x_3 + x_3^2) - x_3x_2^2 - x_2x_3^2}_{\in I}$)

Thm: $\dim(H^*(Fl_n)) = n!$

Geometric basis: Schubert polynomials

$\updownarrow$

$\mathfrak{S}_w$ Schubert classes

Open: Find structure constants for Schuberts, similar to LR rule!

Note: $S_n \curvearrowright R_n = \mathbb{C}[x_1,\dots,x_n]/(e_1,\dots,e_n)$ by permuting $x_1,\dots,x_n$.

Top deg cohomology is sign rep: acts on Vandermonde.

In general: a basis is

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

$$a_i \leq i-1$$

Highest deg: $x_2 x_3^2 x_4^3 \cdots x_n^{n-1} \rightsquigarrow \binom{n}{2}$

Alt top-deg generator:

$$\prod_{i<j} (x_i - x_j) \rightsquigarrow \text{sign rep of } S_n!$$

Want to modify to find other S_n reps as top degree. $\rightsquigarrow$ Springer fibers.

Other S_n-reps: $\mathbb{C}[x_1, x_2]/(e_1, e_2)$ has

basis $\underset{\substack{\downarrow \\ \text{triv. rep}}}{1}$, $\underset{\substack{\downarrow \\ \text{sign rep}}}{x_1 - x_2}$

So $R_2 = V_{\square\square} \oplus V_{\substack{\square \\ \square}}$

<u>Thm</u>: R_n is a graded version of the regular rep ($S_n \curvearrowright \mathbb{C}\langle S_n \rangle$)

Recall: Frobenius map: $\begin{pmatrix} \text{reps of} \\ S_n \end{pmatrix} \to \begin{pmatrix} \text{schur positive} \\ \text{symmetric fns} \end{pmatrix}$

$$\mathrm{Frob}(V_\lambda) = s_\lambda \leftarrow \text{Schur function } (\infty \text{ many vars})$$

$$\mathrm{Frob}(V \oplus W) = \mathrm{Frob}(V) + \mathrm{Frob}(W)$$

Ex: $\mathrm{Frob}\left(V_{(2,1)} \oplus V_{(2,1)} \oplus V_{(1,1)} \oplus V_{(2)}\right) = 2 s_{(2,1)} + s_{(1,1)} + s_{(2)}$

Fact: $\mathrm{Frob}(V \otimes W) = \mathrm{Frob}(V)\,\mathrm{Frob}(W)$

Def: The *graded* Frobenius of a graded S_n-rep $R = \bigoplus_d R^{(d)}$ is the q-analog

$$\mathrm{Frob}_q(R) = \sum \mathrm{Frob}(R^{(d)})\, q^d.$$

Ex: $\mathrm{Frob}_q\left(\mathbb{C}[x_1, x_2]/(e_1, e_2)\right) = s_{(2)} + q\, s_{(1,1)}$

$\uparrow$ graded by deg in x vars

Ex: $\mathrm{Frob}_q\left(\mathbb{C}[x_1, x_2, x_3]/(e_1, e_2, e_3)\right) = s_{(3)} + q\, s_{(2,1)} + q^2 s_{(2,1)} + q\, s_{(1,1,1)}$

Thm (Stanley): $\mathrm{Frob}_q(R_n) = \sum_{T \in SYT(n)} q^{cc(T)} s_{sh(T)}$

(also called maj in SYT case)

Def: The cocharge of a permutation $w = w_1 \ldots w_n$ is computed as follows: label $w_i = 1$ w/ a subscript 0, search to left cyclically for $2, 3, 4, \ldots$ and increment the label if we don't wrap around the end of the word (charge: increment when we do wrap around). Sum of subscripts is $cc(w)$.

Ex: $cc(4_2\, 1_0\, 3_1\, 2_0\, 5_2) = 2+0+1+0+2 = 5$

$\mathrm{charge}(4_1\, 1_0\, 3_1\, 2_1\, 5_2) = 1+0+1+1+2 = 5$

Note: $cc(w) + \mathrm{charge}(w) = \binom{n}{2}$

Def: $cc(T) = cc(\text{reading word}(T))$ for $T \in SYT(n)$.

$\underline{\text{Ex:}}$ $\mathrm{Frob}_q(R_3) = \sum\limits_{T \in SYT(3)} q^{cc(T)} s_{sh(T)}$

T: $\begin{ytableau} 1 & 2 & 3 \end{ytableau}$ $\quad$ $\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$ $\quad$ $\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$ $\quad$ $\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}$

rw: $\quad 1_0 2_0 3_0 \quad 3_1 1_0 2_0 \quad 2_1 1_0 3_1 \quad 3_2 2_1 1_0$

cc: $\quad 0 \quad 1 \quad 2 \quad 3$

$\rightsquigarrow \quad s_{\ydiagram{3}} + q\, s_{\ydiagram{1,2}} + q^2 s_{\ydiagram{1,2}} + q^3 s_{\ydiagram{1,1,1}}$

Springer theory in type A

__Nilpotent cone:__ $\mathcal{N} \subseteq \mathfrak{sl}_n$ nilpotent $n \times n$ matrices

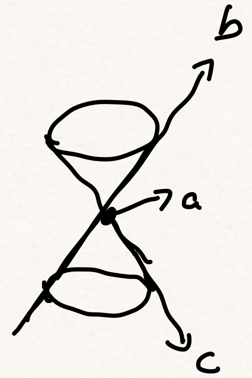
$\underline{\text{Ex:}}$ in $\mathfrak{sl}_2$: $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ nilpotent $\Leftrightarrow$

char poly is t^n (all eigens 0)

$$\left.\begin{array}{c} (a-t)(-a-t) - bc = 0 \\ t^2 - a^2 - bc = 0 \end{array}\right\} \quad \boxed{\begin{array}{l} \text{Want:} \\ a^2 + bc = 0 \end{array}}$$

$(a,b,c) = (0,0,0)$

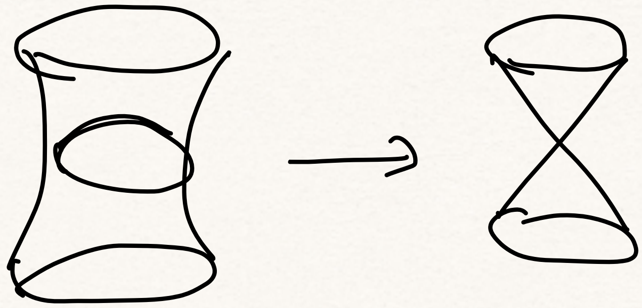
cone shape



Not smooth; <u>resolution of singularities</u> means a map from a smooth space $\tilde{\mathcal{N}}$ to $\mathcal{N}$.

<u>Construction:</u> $\tilde{\mathcal{N}} \subseteq \mathcal{N} \times \mathcal{F}\ell \longrightarrow \mathcal{N}$ by projection.

$$= \{(X, V_\bullet) : XV_i \subseteq V_i \text{ for all } i\}.$$



$\tilde{\mathcal{N}}$ $\qquad\qquad$ $\mathcal{N}$

<u>Def:</u> The <u>springer fiber</u> $\mathcal{B}_X \subseteq \mathcal{F}\ell$ is the fiber of $\tilde{\mathcal{N}} \to \mathcal{N}$ above X, i.e.

$$\mathcal{B}_X = \{V_\bullet \in \mathcal{F}\ell \mid XV_i \subseteq V_i \text{ for all } i\}.$$

<u>Ex:</u> $\mathcal{B}_{\begin{pmatrix}0&0\\0&0\end{pmatrix}} = \mathcal{F}\ell_2$

$$\mathcal{B}_{\begin{pmatrix}0&1\\0&0\end{pmatrix}} = \{V_1 \subseteq \mathbb{C}^2 : \begin{pmatrix}0&1\\0&0\end{pmatrix} V_1 \subseteq V_1\}$$

$$= \{\langle e_1 \rangle\} = pt.$$

$e_2 \to e_1$
$e_1 \to 0$

$$\mathcal{B}_{\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}} = \{\langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle\} = pt.$$

$e_3 \to e_2$
$e_2 \to e_1$
$e_1 \to 0$

Ex: $\mathcal{B}_{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}} = \{\langle e_1+be_3\rangle \subseteq \langle e_1+be_3, e_1+ce_3\rangle\}$ $\quad e_3 \to 0$, $e_2 \to e_1$

$\cup \{\langle e_1+be_3\rangle \subseteq \langle e_1+be_3, e_2+be_3\rangle\}$ $\quad e_1 \to 0$

2-dim'l.

$\mathcal{B}_{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}} = \mathcal{F}\ell_3$

Note: If $gxg^{-1} = y$, $\mathcal{B}_x \cong \mathcal{B}_y$:

$V_\bullet \mapsto gV_\bullet$

$$ygV_\bullet = gxg^{-1}gV_\bullet = gxV_\bullet \subseteq gV_\bullet$$

> So, one distinct springer fiber for each nilpotent conjugacy class!

Conjugacy classes $\leftrightarrow$ Jordan types, all eigens 0

$$\begin{pmatrix}
\boxed{\begin{matrix}0&1&&\\&0&1&\\&&0&1\\&&&0\end{matrix}} &&& \\
& \boxed{\begin{matrix}0&1\\&0\end{matrix}} && \\
&& \boxed{\begin{matrix}0&1\\&0\end{matrix}} & \\
&&& \boxed{0}
\end{pmatrix} \leftrightarrow \text{partition } (4,2,2,1)$$

Write: $\mathcal{B}_\mu$ for partition μ.

Ex: $\mathcal{B}_{(2,1)} = \mathcal{B}_{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}}$ as above.

$\Rightarrow$ One distinct Springer fiber (up to isom) for each partition of n.

$\underline{\text{Thm}}$: $H^*(\mathcal{B}_\mu)$ has an S_n-action,

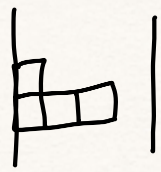
$H^{2d}(\mathcal{B}_\mu) \cong V_\mu$ irred. S_n-rep.

Construction due to Tanisaki, Garsia–Procesi:

$R_\mu := H^*(\mathcal{B}_\mu)$

$\underline{\text{Thm}}$ (Tanisaki): $R_\mu = \mathbb{C}[x_1, \ldots, x_n] \big/ \left(e_r(S) : k - d_k(\mu) < r \leq k \text{ where } k = |S| \right)$

$d_k(\mu) =$ # boxes in last k cols of μ (up to n cols)

Ex: $\mu =$ 

$\underline{k=1}$: $d_k(\mu) = 0$

$1 < r \leq 1$ no good

$\underline{k=2}$: $d_k(\mu) = 1$

$1 < r \leq 2 \Rightarrow r = 2$

$e_2(x_1, x_2), e_2(x_1, x_3), \ldots$

$\Rightarrow x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4$

$\underline{k=3}$: $d_k(\mu) = 2$

$1 < r \leq 3$

$r=2$: $e_2(x_1, x_2, x_3), \ldots$

$r=3$: $e_3(x_1, x_2, x_3), \ldots$

$\underline{k=4}$: $0 < r \leq 4 \rightsquigarrow$ all e_r's of all vars.

Graded Frobenius (Lascoux–Schützenberger)

Thm: $\widetilde{H}_\mu(x;q) = \mathrm{Frob}_q(R_\mu) = \sum_{\substack{T \text{ content} \\ \mu}} q^{cc(T)} s_{sh(T)}$

Def: Cocharge of a word w/ partition

content:

- Search right to left cyclically for $1, 2, 3, \ldots$, label as before
- When highest letter is reached, start over on unlabeled letters
- Repeat until done, sum of subscripts is $cc(w)$

$$cc(1_0 1_0 2_1 3_2 1_0 2_1 1_0 4_2 3_1 3_1 1_0 2_0 2_0) = 8$$

Def: T SSYT, $\quad cc(T) := cc(rw(T))$.

Ex: $\widetilde{H}_{(2,1)}(x;q) = s_{(3)} + q\, s_{(2,1)}$

T: $\boxed{1\ 1\ 2}$ $\qquad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & 1 \\ \hline\end{array}$

$1_0 1_0 2_0 \qquad 2_1 1_0 1_0$