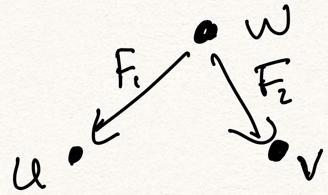
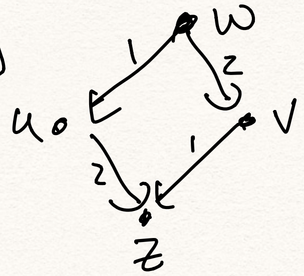
Thm: In any word crystal for sl_n: if



then **Either**:

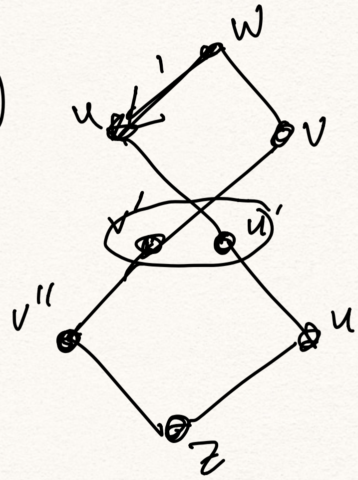
a)



for some z

OR

b)



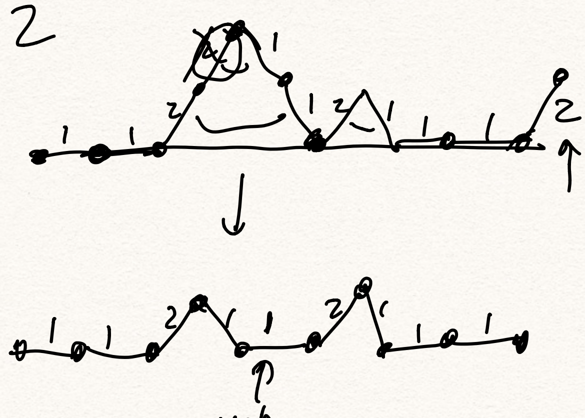
for some v', v'', u', u'', z.

Pf: Sketch Let a = rightmost unbracketed 1 in w (1-2 br)
b = rightmost unbr. 2 in w (2-3 br)

Case 1: Suppose removing b doesn't unbracket a 1.

(Hwk: in a word of 1's and 2's, removing any 2 will unbracket at most one 1)

1 1 2 $\cancel{2}$ 1 1 2 1 1 1 2



Then b is to the right of a,

(if —b—a—, removing b would unbracket a 1.)

$$\text{—}\underset{2}{b}\text{—}\underset{1}{a}\text{—}$$

unbr.

$$w = \text{—}\underset{a}{1}\text{—}\underset{b}{2}\text{—}$$

$$\begin{tikzcd}[column sep=tiny] & w = \text{—}1\text{—}2\text{—} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ \text{—}2\text{—}2\text{—} \arrow[dr, "F_2"'] & & \text{—}1\text{—}3\text{—} \arrow[dl, "F_1"] \\ & \text{—}2\text{—}3\text{—} & \end{tikzcd}$$

still last unbr. 2 in 2-3 pairing

no new unbr

still last unbr 1 b/c we assumed removing b doesn't unbracket a 1

Case 2: Suppose removing b unbrackets a 1, say c = 1.

Subcase 2(a): b is to right of a.

Let d be second-to-last unbr 2 in 2-3 bracketing of $F_1(w)$

$$\text{—}\underset{a}{1}\text{—}\underset{d}{2}\text{—}\underset{b}{2}\text{—}\underset{c}{1}\text{—}$$

?

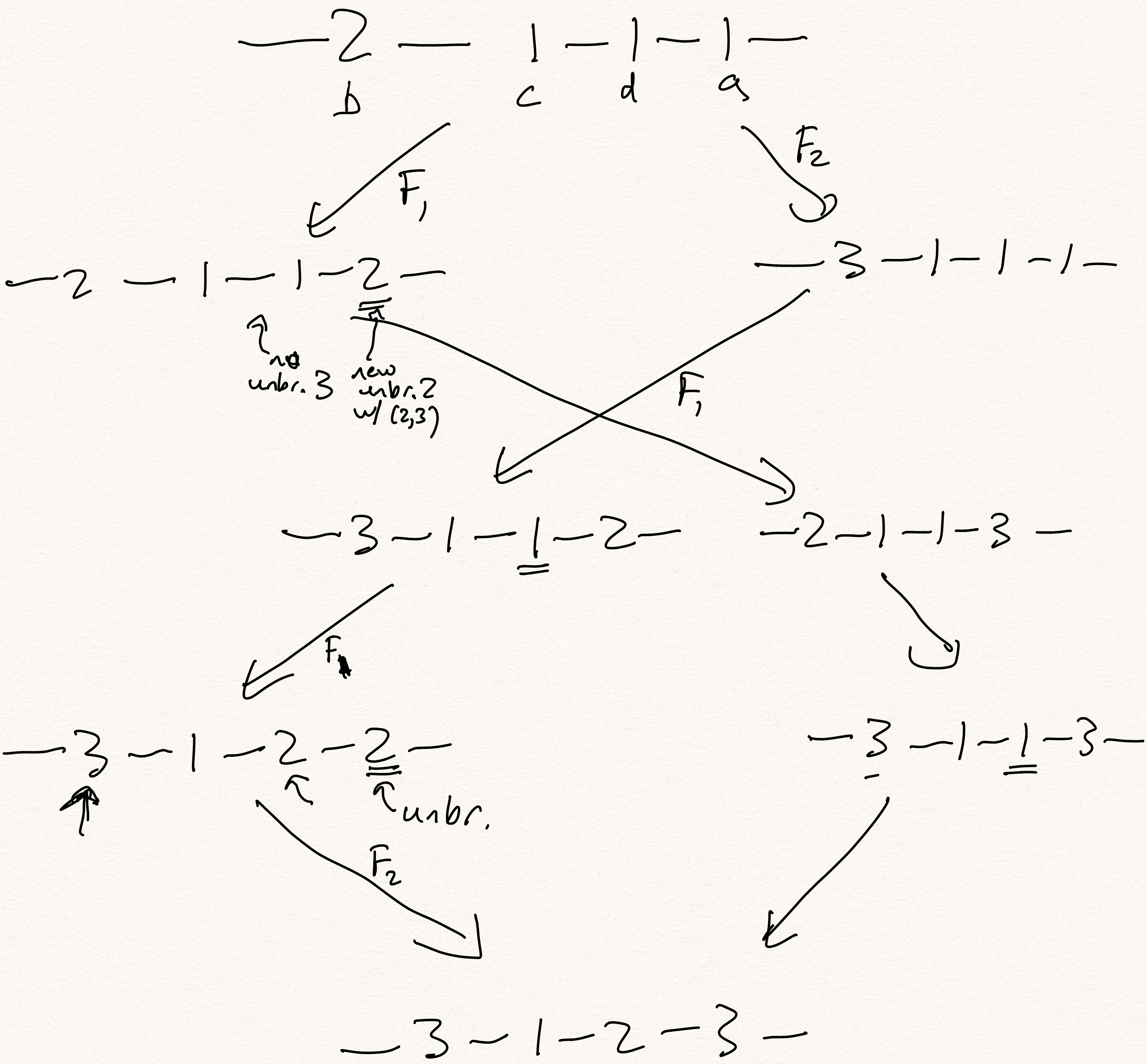
$$\begin{tikzcd}[column sep=tiny] & \text{—}1\text{—}2\text{—}2\text{—}1\text{—} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ \text{—}2\text{—}2\text{—}2\text{—}1\text{—} & & \text{—}1\text{—}2\text{—}3\text{—}1\text{—} \end{tikzcd}$$

$$
\begin{tikzcd}[column sep=small]
 & {} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\
-1-2-3-2- \arrow[d, "F_1"'] & & -2-2-3-1- \arrow[d, "F_2"] \\
-2-2-3-2- \arrow[dr, "F_2"'] & & -2-3-3-1- \ \ \text{unbr.} \arrow[dl, "F_1"] \\
 & -2-3-3-2- &
\end{tikzcd}
$$

Subcase 2(b): b is left of a, there is an unbr. 3 to the left of a.

$$
\begin{tikzcd}[column sep=small]
 & \underset{b}{2} - \underset{c}{1} - (3) - \underset{a}{1} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\
\underset{\substack{\text{last}\\ \text{unbr. } 2}}{2} - 1 - 3 - \underset{\text{br.}}{2} \arrow[dr, "F_2"'] & & -3-1-3-1- \arrow[dl, "F_1"] \\
 & -3-1-3-2 &
\end{tikzcd}
$$

Subcase 2(c): b is left of a but no unbr. 3 to left of a, Let d = 2nd to last unbr. 1 after removing b (possibly $d=c$, d is right of c).



Combinatorial $\mathfrak{sl}_n$-crystals (Type A)

Def: A (type A) Kashiwara crystal (of finite type) is a nonempty set $\mathcal{B}$ along w/ maps

$e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i = 1, \dots, n-1$

$\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$

$\mathrm{wt} : \mathcal{B} \to \Lambda$

"length functions": usually measure # times we can apply e_i, f_i before reaching 0

weight lattice, in this case $\left\{ (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n / (1, 1, \dots, 1) \right\}$

$\alpha_1 L_1 + \alpha_2 L_2 + \dots + \alpha_n L_n$

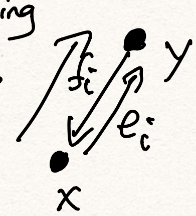
s.t.

K1: If $x, y \in \mathcal{B}$ (nonzero) then $e_i(x) = y$ iff $f_i(y) = x$

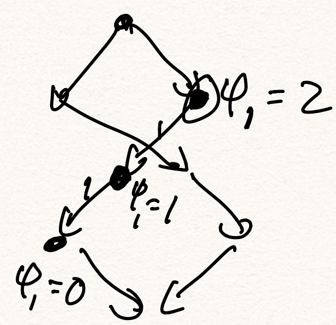
and in this case,

- $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$, $\quad \alpha_i = (0, 0, \dots, \underset{i}{1}, \underset{i+1}{-1}, 0, \dots, 0)$

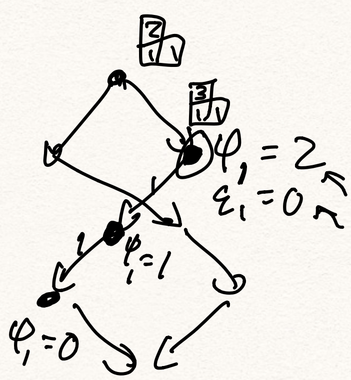
"changing $i+1$ to i"



- $\varepsilon_i(y) = \varepsilon_i(x) - 1$
- $\varphi_i(y) = \varphi_i(x) + 1$



K2: $\varphi_i(x) - \varepsilon_i(x) = \mathrm{wt}(x)_i - \mathrm{wt}(x)_{i+1}$.



$\mathrm{wt} = (2, 0, 1)$

Non-tableau-crystal ex. of a Kashiwara crystal: $(n=2)$

$B = \{v_0, v_{-2}, v_{-4}, \ldots\}$

Need: $e_1, f_1, \varphi_1, \varepsilon_1,$ wt

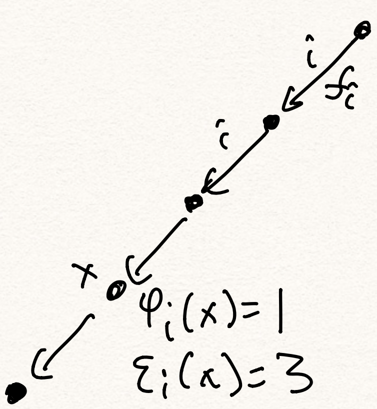
$$\begin{array}{llll}
\bullet\, v_0 & wt=(0,0) & \varphi_1 = 0 & \varepsilon_1 = 0 \\
\downarrow 1 & & & \\
\bullet\, v_{-2} & wt=(-1,1) & \varphi_1 = -1 & \varepsilon_1 = 1 \\
\downarrow 1 & & & \\
\bullet\, v_{-4} & wt=(-2,2) & \varphi_1 = -2 & \varepsilon_1 = 2 \\
\downarrow 1 & & & \\
\bullet\, v_{-6} & wt=(-3,3) & \underline{\varphi_1} = -3 & \varepsilon_1 = 3 \\
\downarrow 1 & & \vdots & \vdots \\
\vdots & & &
\end{array}$$

(Next time: Stembridge axioms)

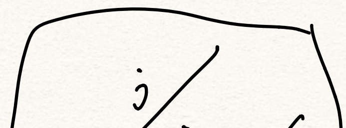
<u>Def</u>: A finite type Kashiwara crystal is called a <u>Stembridge Crystal</u> if it satisfies:

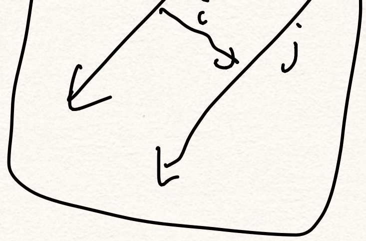
$\boxed{S0}$: (seminormal): ε_i, φ_i measure # times we can apply e_i, f_i respectiely before reaching 0.

i.e. ε_i, φ_i are literally length functions:



$\boxed{S1}$: (length axioms)

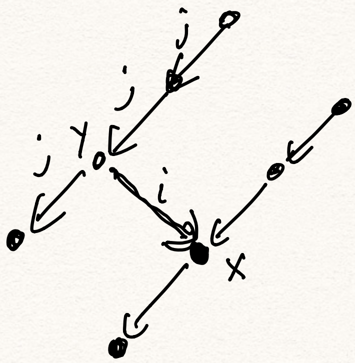




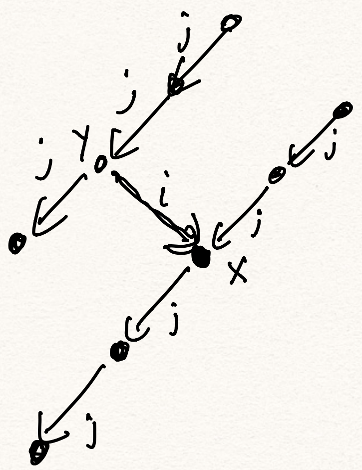
(a) If $|i-j|>1$ and $y=e_i x$

then $\varepsilon_j(y)=\varepsilon_j(x)$

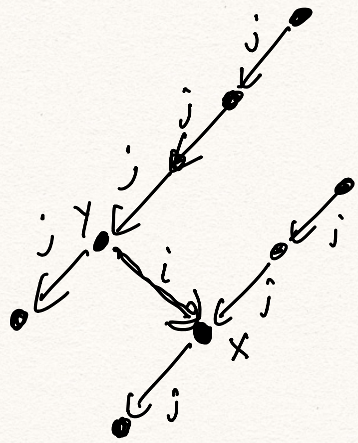
$\varphi_i(y)=\varphi_j(x)$



(b) If $|i-j|=1$, $y=e_i(x)$, then either:



OR



i.e. either:

$\varphi_j(x)=\varphi_j(y)+1$ OR $\varphi_j(x)=\varphi_j(y)$

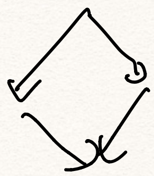
and $\varepsilon_j(x)=\varepsilon_j(y)$ and $\varepsilon_j(y)=\varepsilon_j(x)+1$.

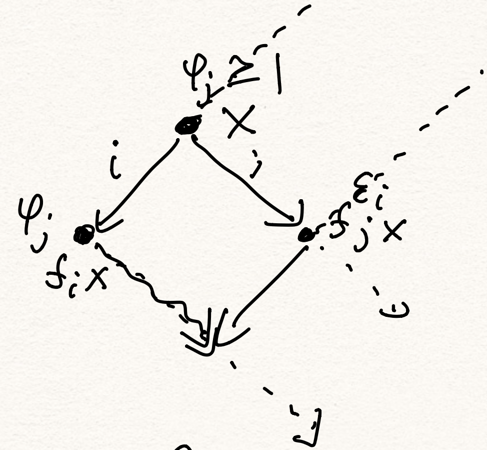
<u>Hwk</u>: Prove the length axioms hold for word crystals.

$\boxed{S2}$ If $x\in B$ w/ $\varphi_j(x)>0$, $\varphi_i(x)>0$, and $\varphi_j(f_i x)=\varphi_j(x)$, then:

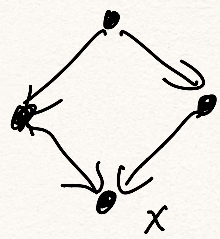
$f_i f_j x = f_j f_i x \neq 0$

and $\boxed{\varepsilon_i(f_j x) = \varepsilon_i(x)}$





$\boxed{\text{Dual S2}}$: reverse roles of $e \leftrightarrow f$, $\varphi \leftrightarrow \varepsilon$ in S2



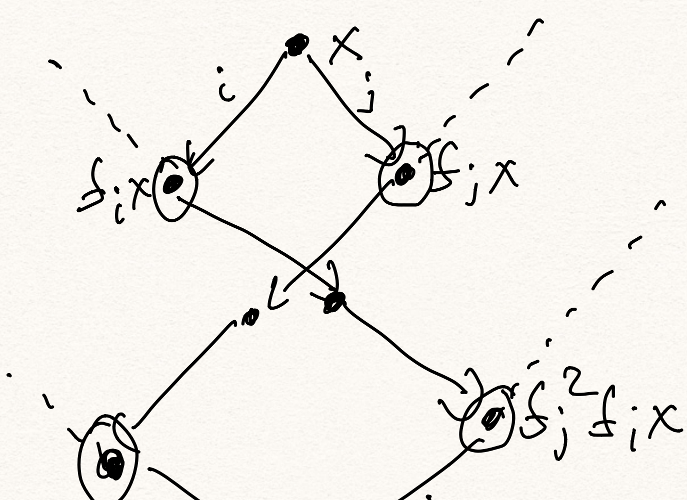
$\boxed{\text{S3}}$ If $x \in B$ and $\varphi_i(x), \varphi_j(x) > 0$,

$$\left.\begin{aligned} \varphi_j(f_i(x)) &= \varphi_j(x) + 1, \\ \varphi_i(f_j(x)) &= \varphi_i(x) + 1 \end{aligned}\right\},$$

then: $f_j f_i^2 f_j x = f_i f_j^2 f_i x \neq 0$,

$$\varepsilon_i(f_j x) = \varepsilon_i(f_j^2 f_i x)$$

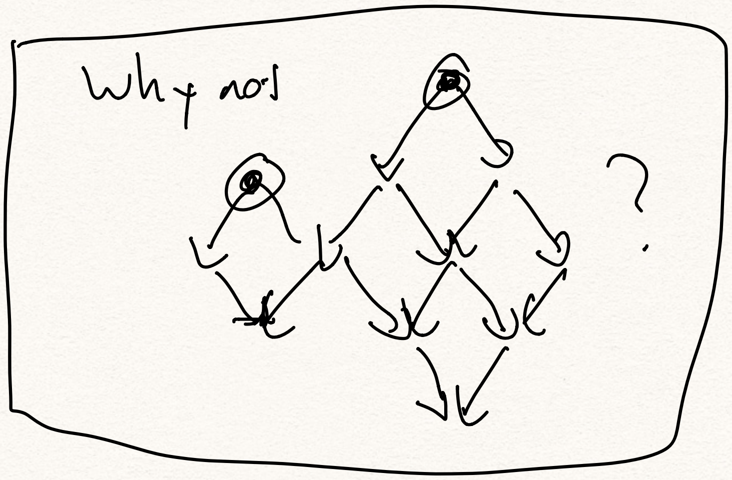
and $\varepsilon_j(f_i x) = \varepsilon_j(f_i^2 f_j x)$



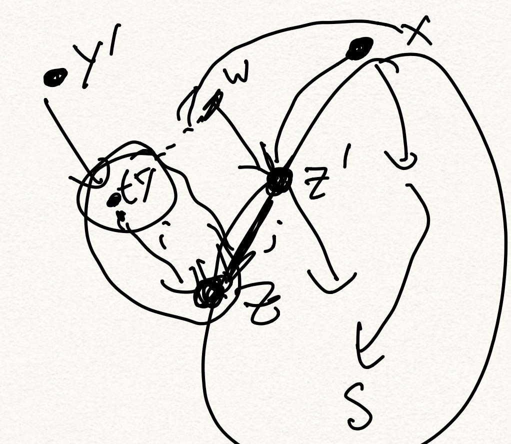
$f_j f_i^2 f_j x = f_i f_j^2 f_i x$

$\boxed{\text{dual S3}}$ Replace $e_i \leftrightarrow f_i$, $\varepsilon_i \leftrightarrow \varphi_i$ for all i.

<u>Thm</u>: Connected Stembridge crystals have unique highest weight elts (killed by all e_i)



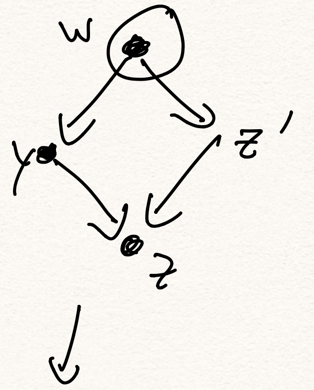
<u>Pf idea</u>: Assume x is highest weight but not all $y \in B$ are "below" x (reachable from x by f_i operators)



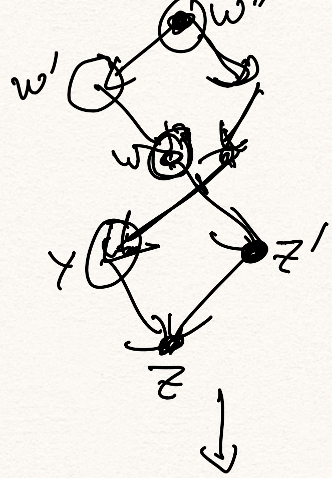
Consider z maximal in S below x s.t. z is <u>covered by</u> ($f_i y = z$ for some i) some y which is not below x.

Since $z \in S$, and $z \neq x$ since x is maximal, $\exists z' \in S$ s.t. $z' \xrightarrow{j} z$ for some j.

So by dual S2, dual S3, either:



OR



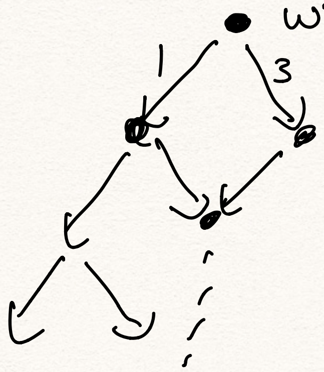
In this case, w must be in S, so $y \in S$, $\rightarrow\leftarrow$.

In this case, $w \in S$, similarly $w' \in S$, similarly $w'' \in S$. Thus $y \in S$, $\rightarrow\leftarrow$.

QED.

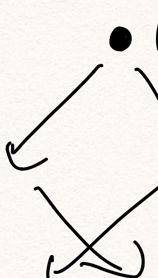
<u>Thm</u>: If two conn. Stemb. crystals have the same highest weight ($wt(x)$), they are isomorphic.

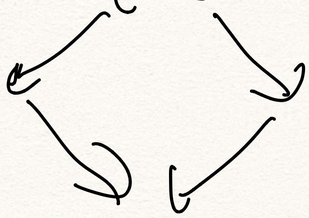
<u>Pf idea</u>:



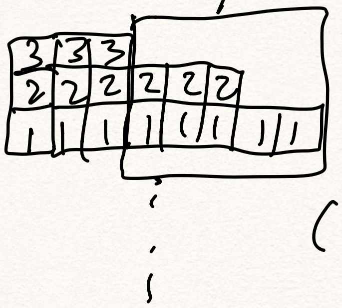
wt $\Rightarrow$ some info about φ_i, ε_i

At each step keep track of φ_i, ε_i using S1, S2, S3.





<u>Cor:</u> Every connected Stembridge crystal is a crystal of tableaux.



(highest weight elt).

Weights: $\mathbb{Z}^n/(1,1,1,\dots,1)$

$(8,6,3)\sim(7,5,2)$ for $n=3$

$\sim(\underbrace{5,3}_{},0)$

$n=5$: $(2,3,3,0) \longleftrightarrow (\underbrace{8,6,3,0,0}_{})\sim(9,7,4,1,1)$

length $n-1$ vector

length n equiv. class of

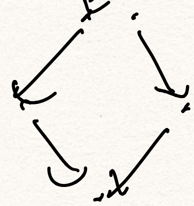
<u>Consequence:</u> To show some symmetric function

$$\sum_{P\in S} x^P \qquad \text{in vars } x_1,\dots,x_n$$

is Schur positive, suffices to define operators $\underline{e_i, f_i}$ for $i=1,\dots,n-1$ on S $(S\to S\sqcup\{0\})$ and wt/length functions wt, ε_i, φ_i that satisfy the Kashiwara and Stembridge axioms.

$s_{(2,0,0)}$

$s_{(2,0,0)}$



$s_{(2,1,0)}$

$\Rightarrow$ fn is $\underline{s_{(2,0,0)}} + \underline{s_{(2,0,0)}} + s_{(2,1,0)}$

[Done for Stanley symmetric functions and ordered set partitions].

(Found Schur function expansions).