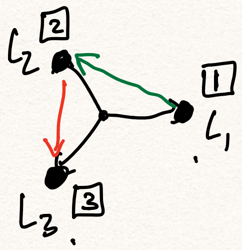
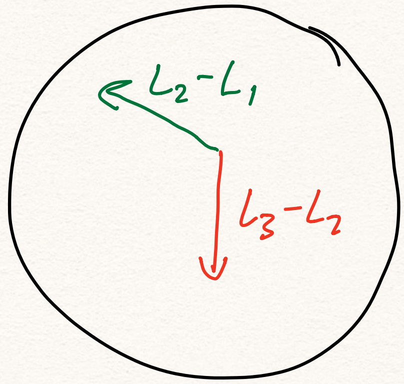
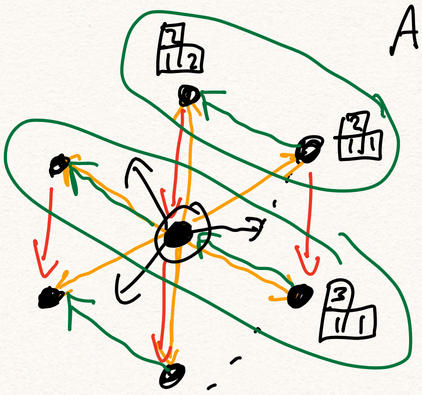


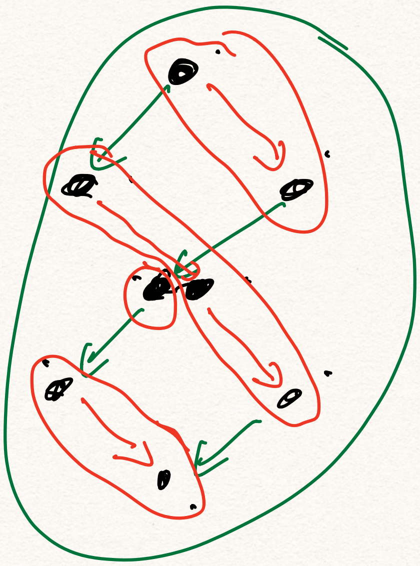
Triv rep of sl_3

$\underline{sl_3} \longrightarrow \underline{gl_3}$



Adjoint rep of sl_3





$8 = 3^2 - 1$

---

sl_n-representation theory

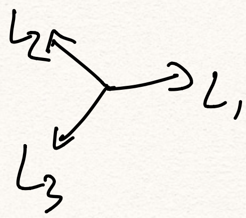
- $sl_n(\mathbb{C}) = \{ \underline{X \in Mat_n(\mathbb{C}) : tr(X) = 0} \}$ $\quad \begin{pmatrix} x_{11} & x_{12} & x_{13} & \\ & & \ddots & \\ & & & x_{nn} \end{pmatrix}$ n^2 vars, $tr = 0$
- $\underline{\underline{\mathfrak{h}}} \subseteq sl_n(\mathbb{C})$ Cartan, diagonal matrices
- $\mathfrak{h}^* = \{ \text{maps } \mathfrak{h} \to \mathbb{C} \}$
- <u>Weights</u> of a rep V of sl_n:
  joint eigenvalues of $\mathfrak{h}$.
  $(\alpha \in \mathfrak{h}^*)$ $\quad H v = \underline{\alpha}(\underline{H}) v$ for any $H \in \mathfrak{h}$.
- $\underline{\underline{V_\alpha}} = \{ v \in V : Hv = \alpha(H) v \text{ for all } H \in \mathfrak{h} \}$
- <u>Decomposition</u> (for finite-dim. rep V)

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$$

finite sum.

- <u>Weight lattice</u>: $\mathbb{Z}$-lattice gen by $L_1, \dots, L_n$: $L_i \begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{pmatrix} = x_i$.

$$L_1 + L_2 + \cdots + L_n = 0$$



- General wt $\alpha = \underbrace{\alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_n L_n}_{\text{integers.}}$

- <u>Adjoint rep</u>: $\mathfrak{sl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}(\mathfrak{sl}_n(\mathbb{C}))$

dim of adj rep $= \dim(\mathfrak{sl}_n(\mathbb{C})) = n^2 - 1$

<u>basis</u>: • $H_1, H_2, \dots, H_{n-1}$ $\Big\}\ n-1$

$$H_i = \begin{array}{c} \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \leftarrow i+1 \\ \\ \\ \end{array}$$

• E_{ij} for $i \neq j$ $\Big\}\ n^2 - n$

$$= \begin{array}{cc} \begin{array}{cccc} & & j & \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \\ \leftarrow i \\ \\ \\ \end{array} \end{array}$$

$$[\underline{H}, \underline{E_{ij}}] = H \cdot E_{ij} - E_{ij} \cdot H \qquad H = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

$$= \underline{(x_i - x_j)}\, \underline{E_{ij}}.$$

i.e. $\underline{E_{ij}}$ spans weight space of weight

$L_i - L_j$ $\qquad (L_i - L_j)(H) = x_i - x_j$

If W is adj. rep $\boxed{W = \mathfrak{h} \oplus \bigoplus W_{L_i - L_j}}$

- **Roots**: weights of adjoint rep

  $\Rightarrow$ Roots of $\mathfrak{sl}_n$ are $\{\underline{L_i - L_j} : i \neq j\}$

- **Positive roots**: $\underline{L_i - L_j}$: $i < j$

  Negative roots: $L_j - L_i$: $i < j$

(inner product w/ vector $(n, n-1, \ldots, 1)$)

$\bullet\ (0\,0\,0\,1\,0\,0\,{-1}\,0\,0)$

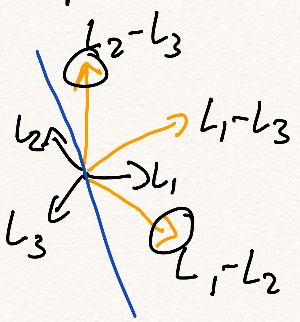
$L_i - L_j$

0

$(\ \ -1 \ \ 1\ \ )$

$-L_i + L_j$

- **Simple roots**: defined as positive roots that are not positive linear combinations of other positive roots.



$L_1 - L_3 = (L_1 - L_2) + (L_2 - L_3)$

not simple $\qquad$ simple

$$L_{\underline{4}} - L_{\underline{7}} = \underline{(L_4 - L_5)} + \underline{(L_5 - L_6)} + \underline{(L_6 - L_7)}$$

$$\boxed{\text{Simple roots for } \mathfrak{sl}_n : \underline{\alpha_i := L_i - L_{i+1}} \text{ for } i = 1, \ldots, n-1}$$

$\leftrightarrow \{E_{i,i+1}\}$, $\mathfrak{h}$ generate all of $\mathfrak{sl}_n$ as a Lie alg (using $+$, scalar$\cdot$, $[\ ,\ ]$)

Ex: $[E_{i,i+1}, E_{i+1,i+2}] = c\, E_{i,i+2}$

Thm: Let V be rep of $\mathfrak{sl}_n$, $v_\alpha \in V$ is a weight vector for V of weight α.

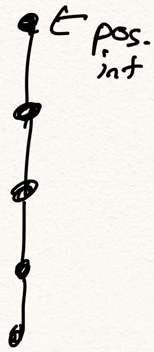
Then: $E_{ij} v_\alpha$ is a wt vector of weight $\alpha + (L_i - L_j) \in \mathfrak{h}^*$

(In particular $E_{i,i+1} v_\alpha$ is wt vector of weight $\underbrace{\alpha + (L_i - L_{i+1})}_{\boxed{\alpha + \alpha_i}}$)

$\alpha_1 L_1 + \cdots + \alpha_n L_n$
$(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \vdots\ \alpha_{n-1} - \alpha_n)$

Pf: (HWK exercise on next hw).

Cor: Can decompose V into $\mathfrak{sl}_2$-chains in each α_i direction.

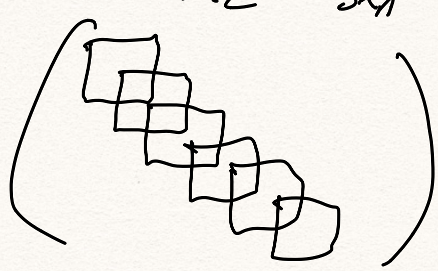


$H_i = \begin{pmatrix} \boxed{\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}} \end{pmatrix}$

$E_{i,i+1} = \begin{pmatrix} \boxed{\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}} \end{pmatrix}$

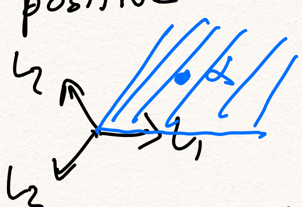
$E_{i+1,i} = \begin{pmatrix} \boxed{\begin{smallmatrix} 0 & \\ 1 & 0 \end{smallmatrix}} \end{pmatrix}$

generate $\mathfrak{sl}_2 \subseteq \mathfrak{sl}_n$

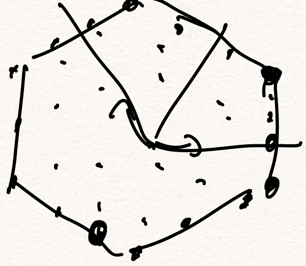


Irred. rep of $\mathfrak{sl}_n$:

Formed by starting w/ some "highest wt" α (positive w.r.t each $L_i - L_{i+1}$)



and then close under all required $\mathfrak{sl}_2$-chains.



"hexagon"

Def: Weyl group of $\mathfrak{g}$: group generated by reflections about hyperplanes orthogonal to roots in $\mathfrak{h}^*$

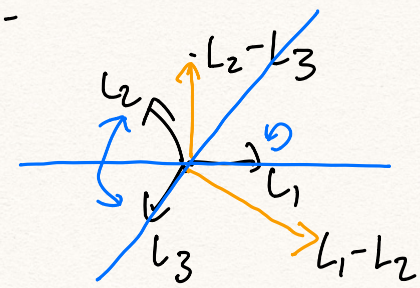
Ex: Weyl gp of sl_n:

Reflect about $L_i - L_j$

$\alpha = \alpha_1 L_1 + \cdots + \alpha_n L_n$

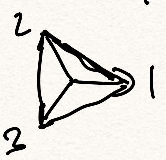
$$\downarrow$$

$\alpha_1 L_1 + \cdots + \alpha_j L_i + \cdots + \alpha_i L_j + \cdots + \alpha_n L_n$



Claim: Reflection about $L_i - L_{i+1}$ switches L_i, L_{i+1} (transp. $(i\ \ i+1)$)

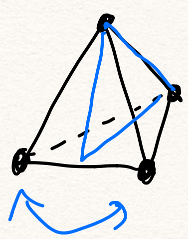
(symmetry gp of



is S_3)

$\boxed{\text{Claim: Weyl gp is } S_n}$

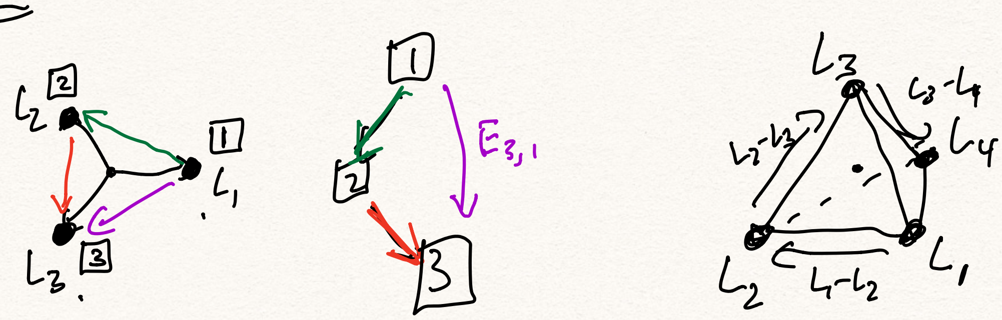
Next time: Words, bracketing, tableaux for sl_n.



Word crystals for sl_n:

$$V^{\substack{(1,\,0,\,0,\,0,\,\ldots,\,0)}}_{\alpha_1\ \alpha_2\ \cdots\ \alpha_n}$$

V^{L_1}: irred. rep w/ highest weight L_1



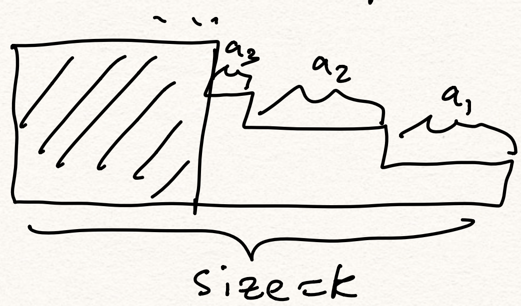
V^{L_1}:

$$\begin{tikzcd}[row sep=large]
\boxed{1} \arrow[d, "F_1 := E_{2,1}"] \\
\boxed{2} \arrow[d, "F_2 := E_{3,2}"] \\
\boxed{3} \arrow[d, "F_3 := E_{4,3}"] \\
\vdots \arrow[d, "F_{n-1} := E_{n,n-1}"] \\
\boxed{n}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{array}{c} \mathfrak{sl}_2\text{-chain:} \\
\begin{tikzcd}
\bullet \arrow[d] \\ \bullet \arrow[d] \\ \bullet \arrow[d] \\ \bullet \arrow[d] \\ \bullet
\end{tikzcd}
\end{array}$$

Thm: $\left(V^{L_1}\right)^{\otimes k} = \bigoplus c_{(a_1,\dots,a_{n-1})} V^{(a_1,\dots,a_{n-1})}$

$\alpha_1-\alpha_2 \quad \alpha_2-\alpha_3 \quad \alpha_{n-1}-\alpha_n$

$c_{(a_1,\dots,a_{n-1})} = \#$ SYT of shape



(Proof is same as $\mathfrak{sl}_3$ using RSK)

Bracketing rules: to compute $F_i = E_{i+1,i}$ on tensor word $\boxed{w_1} \otimes \boxed{w_2} \otimes \cdots \otimes \boxed{w_k} =: w_1 \cdots w_k$

- Bracket all $i+1$'s w/ i's, change
  $($ $\qquad\qquad$ $)$

last unpaired i to $i+1$.

- $E_i = E_{i,i+1}$: bracket $i+1$'s w/ i's, change <u>leftmost unpaired $i+1$</u> to i.

<u>Cor</u>: A word w is <u>highest weight</u> if it's ballot for $i, i+1$-subword for all i.

<u>Fact</u>: RSK ins, reading words don't change crystal structure (as before)

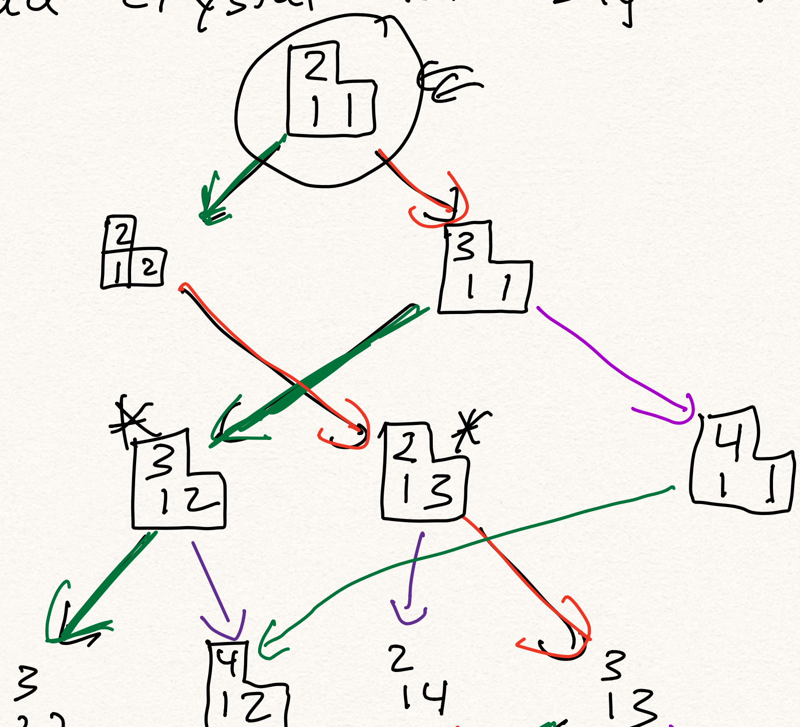
Q: what does $E_{i,j}$ do to a word?

Ex: $E_{1,3}$ or $E_{3,1}$ in sl_3:

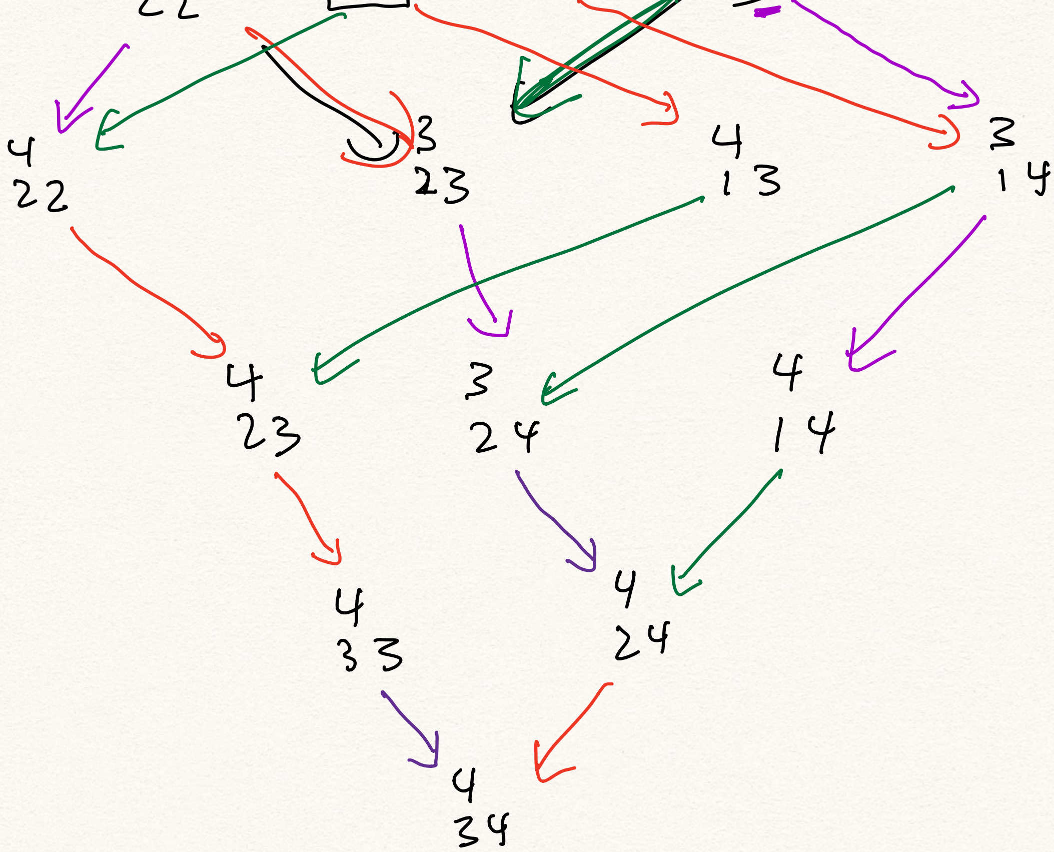
$E_{3,1} = [E_{3,2}, E_{2,1}]$

Not sure!

<u>Ex</u>: Tableau crystal for sl_4 of shape ⊞.



Every SSYT of shape ⊞ in 1,2,3,4 appears here!



Facts: (1) Character of tableau crystal for $\mathfrak{sl}_n$ of shape λ is

$$s_\lambda(x_1, x_2, \ldots, x_n)$$

(2) Littlewood-Richardson rule:

$$s_\lambda(x_1, \ldots, x_n) \cdot s_\mu(x_1, \ldots, x_n) = \sum c^\nu_{\lambda\mu} \, s_\nu(x_1, \ldots, x_n)$$

where $c^\nu_{\lambda\mu}$ = # pairs of SSYT in letters $1, \ldots, n$ of shapes λ, μ whose concatenated reading word is ballot.

[same reasoning)

Stembridge axioms (2004)

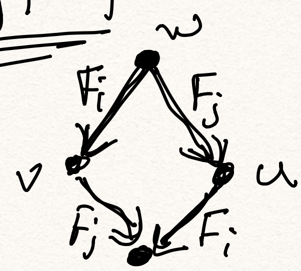
How to know if a graph is a tableau crystal for sl_n?

Properties of tableau crystal graphs:

① F_i, F_j commute for $|i-j|>1$

sl_3: F_1, F_2 X

sl_4: F_1, F_2, F_3



Ex: F_1, F_3
brackets 1,2
brackets 3,4

i.e. if $F_i(w)=v$ and $F_j(w)=u$

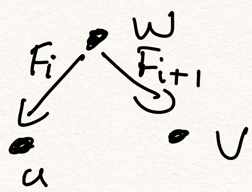
then $F_j(v)=F_i(u)$ and nonzero

(Similarly E_i commutes w/ E_j)

$(F_i F_j = F_j F_i)$

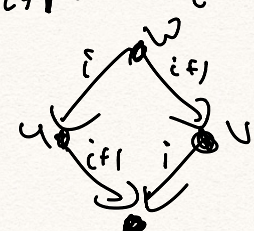
② If $F_i w=u$,

$F_{i+1} w=v$

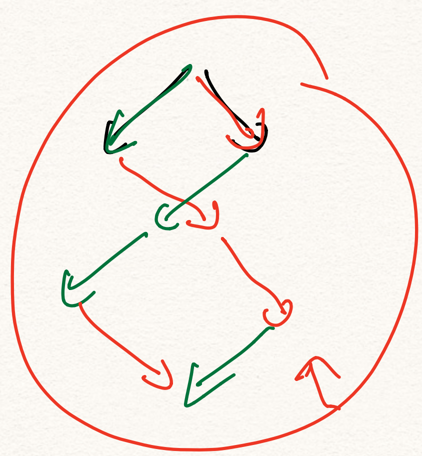


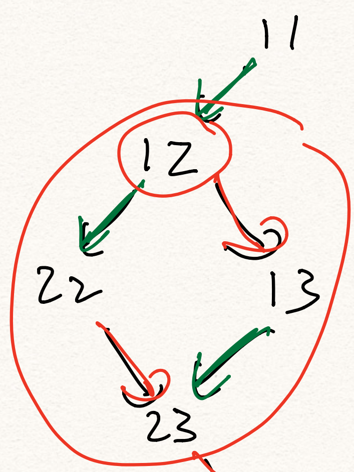
then either:

(a) $F_{i+1} u = F_i v \neq 0$



$F_{i+1} u = F_i v$





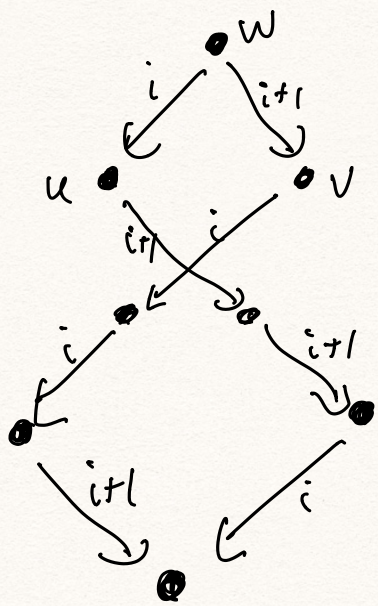
OR:

(b) $F_{i+1}^2 u \neq 0$,

33

$$F_i^2 v \neq 0, \quad F_i v \neq F_{i+1} u$$

$$F_i F_{i+1}^2 u = F_{i+1} F_i^2 v \neq 0$$



Next time: prove that bracketing satisfies property 2

then: State Stembridge axioms, show they uniquely determine tableau crystal graphs.