$\mathfrak{sl}_3 = \{ X \in Mat_3(\mathbb{C}) : tr(X) = 0 \}$

<u>Goal</u>: Analyze rep thy of $\mathfrak{sl}_3$, and all semisimple Lie algebras $\mathfrak{g}$.

$\mathfrak{sl}_3 : X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ $\boxed{x_{11} + x_{22} + x_{33} = 0}$

<u>Basis</u>: $H_{12} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$, $H_{23} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

E_{ij} = matrix w/ 1 in (i,j) position
0's everywhere else.

$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ...

<u>Note</u>:
- Subspace gen. by H_{12}, E_{12}, E_{21} is a copy of $\mathfrak{sl}_2$
- " " " H_{23}, E_{23}, E_{32}
- " " " H_{13}, E_{13}, E_{31}

$\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

Common eigenvectors of H_1, H_2 : <u>weight space</u> of a representation.

<u>Def:</u> $\mathfrak{h} \subseteq sl_3(\mathbb{C})$ subspace of diag matrices
$\mathfrak{h} = \langle H_1, H_2 \rangle$

<u>Def:</u> A <u>cartan subalgebra</u> of a Lie alg. $\mathfrak{g}$ is a maximal abelian subalg. that acts diagonalizably $\boxed{\text{adjoint rep}}$.
$\mathfrak{h}$ $[H, E] = 2E$ $[H, F] = -2F$

<u>Fact:</u> Everything in $\mathfrak{h}$ acts diagonalizably <u>on V</u> for any rep. V of $\mathfrak{g}$.

<u>Fact:</u> Commuting diagonalizable matrices are <u>simultaneously diagonalizable</u>. (i.e. they have same 1-d eigenspaces)

$\Rightarrow$ $V = \bigoplus V_\alpha$ where V_α is eigenspace of $\mathfrak{h}$ w/ "joint eigenvalue" α.

<u>Def:</u> A <u>joint eigenvalue</u> for $\mathfrak{h}$ acting on V is a function $\alpha \in \mathfrak{h}^* = \{\mathfrak{h} \xrightarrow{\text{linear}} \mathbb{C}\}$ s.t. $\exists v_\alpha \in V$ s.t. $H v_\alpha = \alpha(H) v_\alpha$ for any $H \in \mathfrak{h}$.

<u>Ex:</u> In sl_3: represent α as
$$\boxed{(\alpha_1, \alpha_2) = (\alpha(H_{12}), \alpha(H_{23}))}$$

Nicer: $\underline{(\alpha_1, \alpha_2, \alpha_3)} = (\alpha(H_{12}), \alpha(H_{23}), \alpha(H_{13}))$

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

Note: $H_{12} + H_{23} - H_{13} = 0 \implies \boxed{\alpha_1 + \alpha_2 + \alpha_3 = 0}$

The space $\mathfrak{h}^*$ gen. by:

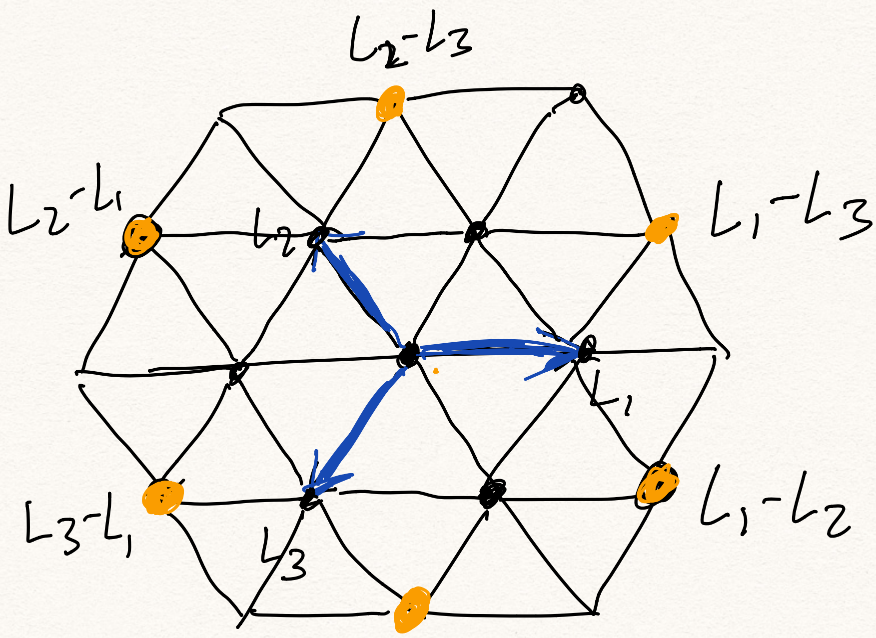
$$L_1 : \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \to x_1$$

$$L_2 : \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \to x_2$$

$$L_3 : \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \to x_3$$

Then $L_1 + L_2 + L_3 = 0$

Picture:



Thm: The weights of the adjoint rep. of sl_3 are $\boxed{\pm(L_1-L_2),\ \pm(L_1-L_3),\ \pm(L_2-L_3)}$

Pf: $sl_3 \xrightarrow{ad} gl(sl_3)$

$X \mapsto [X,-]$

$$H = \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_3 \end{pmatrix} \in \mathfrak{h}$$

$$[H, E_{12}] = H \cdot E_{12} - E_{12} \cdot H$$
$$= x_1 E_{12} - x_2 E_{12}$$
$$= \boxed{(x_1 - x_2) E_{12}}$$

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In general $[H, E_{ij}] = (x_i - x_j) E_{ij}$

$\Rightarrow$ $L_i - L_j$ is the eigenvalue in $\mathfrak{h}^*$

QED

Cor: sl_3 decomposes as

$$\mathfrak{h} \oplus V_{L_1-L_2} \oplus V_{L_2-L_1} \oplus \cdots$$

Def: The nonzero weights of $ad(\mathfrak{g})$ are called roots of $\mathfrak{g}$.

$R = \{\text{roots of } \mathfrak{g}\}$

Can write $\boxed{\mathfrak{g} = \underbrace{\mathfrak{h}}_{\text{weight } 0} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha}$

- $\mathfrak{g}_\alpha$ is a root space of weight α

$\shortparallel$

$\{V \in \mathfrak{g} : [H, V] = \alpha(H) V$ for all $H \in \mathfrak{h}\}$.

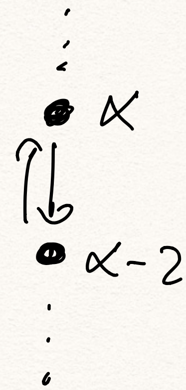
<u>Def:</u> Λ_R = lattice in $\mathfrak{h}^*$ gen. by R

<u>Ex:</u> In $\mathfrak{sl}_2$: $R = \{2, -2\}$

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad [H, E] = 2E$$
$$[H, F] = -2F$$

$\alpha(H) = 2$

$\alpha(cH) = 2c$



<u>Thm:</u> (Weights all differ by elts of Λ_R)
For a rep V of Lie alg $\mathfrak{g}$, let $V = \bigoplus V_\beta$ be its decomp. into weight spaces for $\mathfrak{h} \subseteq \mathfrak{g}$. Then if $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$ then $Xv \in V_{\beta+\alpha}$.

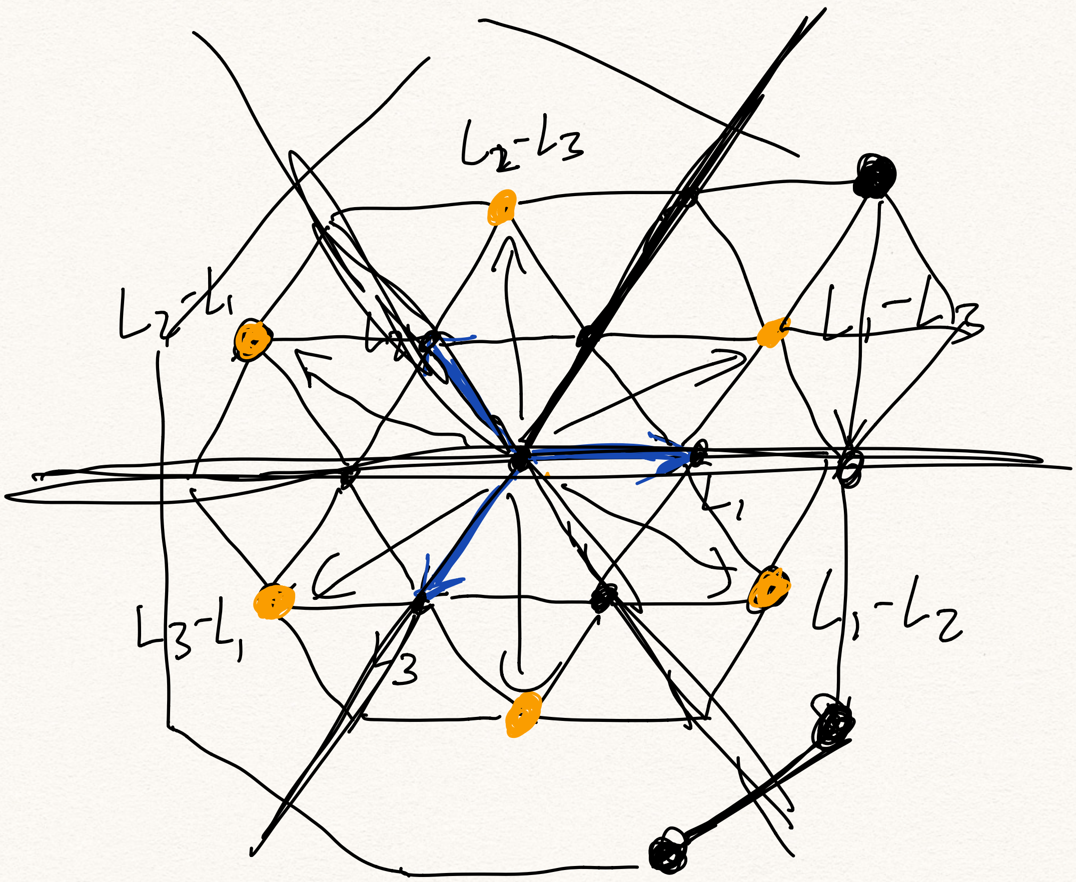
<u>Pf:</u> Let $H \in \mathfrak{h}$. Then

$$\begin{aligned} HXv &= ([H, X] + XH)v \quad (X \in \mathfrak{g}_\alpha) \\ &= (\alpha(H)X + XH)v \\ &= \alpha(H)Xv + XHv \quad (v \in V_\beta) \\ &= \alpha(H)Xv + \beta(H)Xv \end{aligned}$$

$$= \underline{(\alpha(H) + \beta(H))} X_v$$

Therefore $\quad X_v \in V_{\alpha+\beta}.$ QED.

Picture:



Root systems

$\mathfrak{sl}_3(\mathbb{C})$ representations

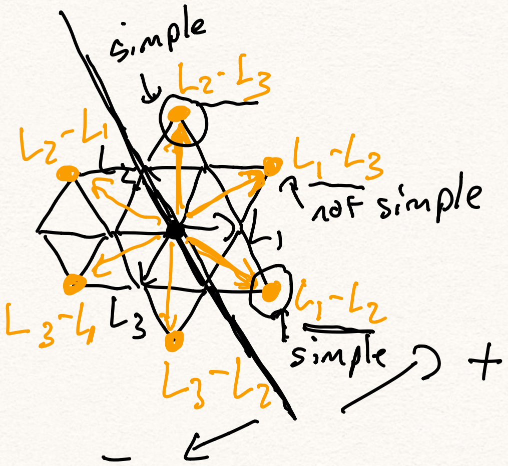
Roots are weights of adjoint representation

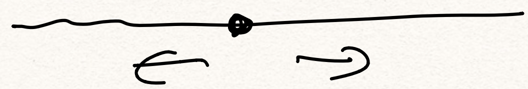
$$\mathfrak{h} = \left\{ \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_3 \end{pmatrix} : x_1+x_2+x_3=0 \right\}$$

$\mathfrak{h}^*$ gen. by

$L_1\begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} = x_1$, $L_2(X) = x_2$, $L_3(X) = x_3$

$$R = \{\pm(L_1-L_2),\ \pm(L_2-L_3),\ \pm(L_1-L_3)\}$$





Last time: If V is rep of $\mathfrak{sl}_3(\mathbb{C})$

$V \cong \bigoplus V_\alpha$ and if $v_\alpha \in V_\alpha$

then

$$\boxed{E_{12} v_\alpha \in V_{\alpha+(L_1-L_2)}}$$

$$\underbrace{\begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}} \rightarrow \boxed{E_{23} v_\alpha \in V_{\alpha+(L_2-L_3)}}$$

$$\boxed{E_{13} v_\alpha \in V_{\alpha+(L_1-L_3)}}$$

$E_{13} = [E_{12}, E_{13}]$

(Also have E_{21}, E_{32}, E_{21})

$$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ 1 & & 0 \end{pmatrix}$$

F_{12}, F_{23}, F_{12}

$$\boxed{\text{In general for any root system } R \text{, choose a hyperplane to separate } R \text{ into } R_+ \cup R_-}$$

Def: A highest weight vector in a rep V of $\mathfrak{sl}_3$ is a vector $v_\alpha \in V_\alpha$ (for some weight α) s.t. $E_{12} v_\alpha = E_{23} v_\alpha = E_{13} v_\alpha = 0$

(In general case: killed by all positive root vectors)

Def: A simple root is a positive root (+) that is not the positive sum of other positive roots.

$$(\alpha_1, \alpha_2) = \left(\alpha\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \alpha\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right)$$

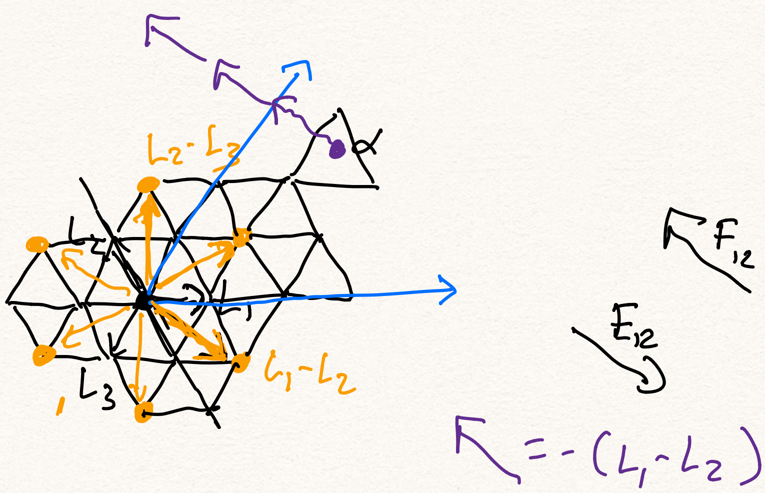
To find a highest weight vector, start w/ any nonzero vector $v \in V$ keep applying E_{12}, E_{23} until you can't apply either. (May not be unique!)

Fact: Irreducible sl_3-reps have a unique highest weight vector.

Q: What do sl_3 reps w/ unique highest weight vector v_α look like?

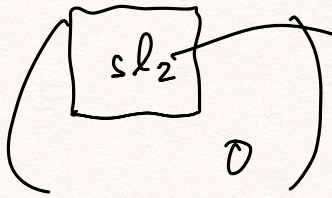
Thm: α is on lattice gen. by L_1, L_2, L_3.

Pf: Look at sl_2 chain starting from α in $L_1 - L_2$ direction, $L_2 - L_3$ direction.



i.e. if V rep of sl_3, it's a rep of sl_2 as well

$sl_2 \subseteq sl_3$

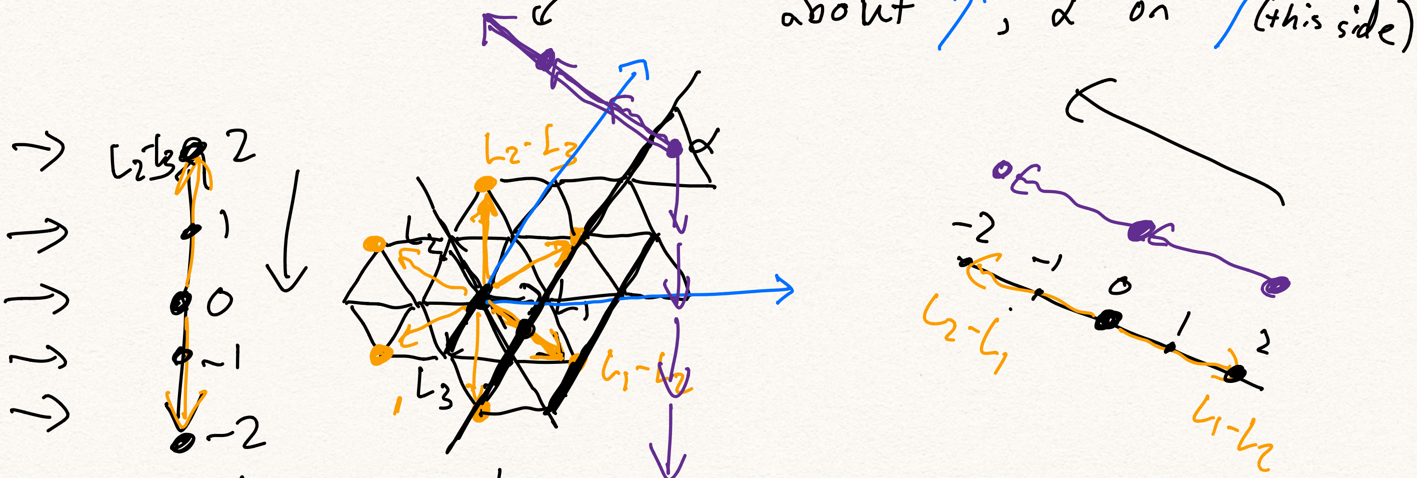


root is $L_1 - L_2$

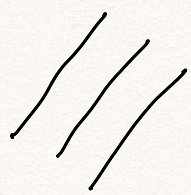
Moreover, α is highest weight of one of the sl_2 chains that V decomposes into.

Ignoring L_3 component:

this purple chain is symmetric about ↗, α on ↗ (this side)

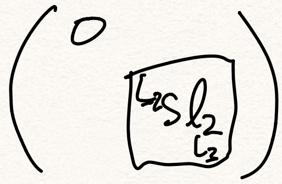


By sl_2-rep theory, α is on one of the black lines



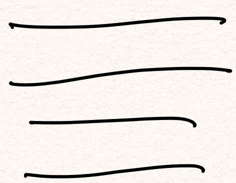
(in L_3 direction)

Also look at V as a



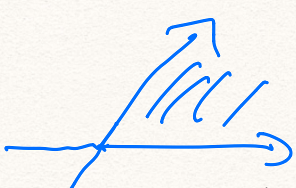
rep.

α is on black lines



$\Rightarrow$ α lies on lattice. QED.

<u>Cor:</u> α is in



because α is on lower side of

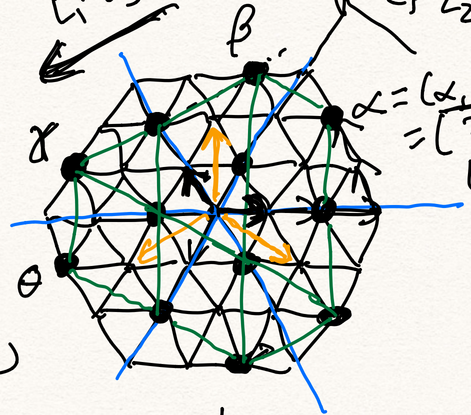
and

above

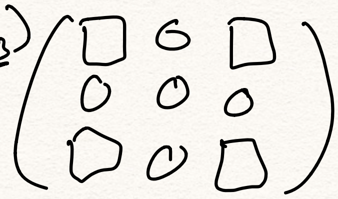
subtract L_1-L_3 subtract L_1-L_2

Ex:

subtract L_2-L_3



$\alpha = (\alpha_1-\alpha_3, \alpha_2-\alpha_3) = (3,2)$
$\qquad\quad L_1 \quad L_2$



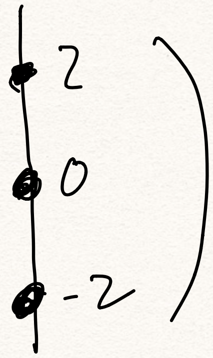
sl_2

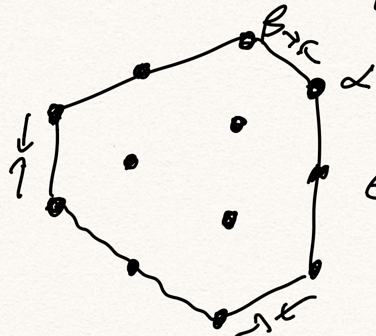
β is highest wt in L_1-L_3 sl_2-direction (because otherwise α isn't highest wt)

$\alpha = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3$
$\;\; = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3(-L_1-L_2)$
$\;\; = (\alpha_1-\alpha_3)L_1 + (\alpha_2-\alpha_3)L_2$

Final picture: Hexagon w/ all internal pts in triangular grid.

(analog of

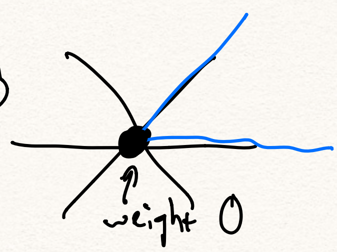




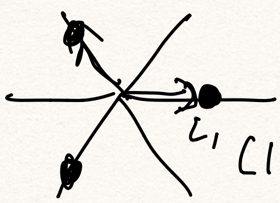
dim of this is 12

this set of weights gives an irred. sl_2-rep.

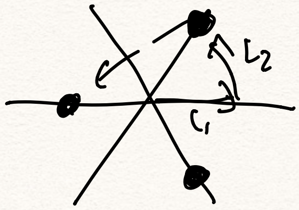
Ex: ①



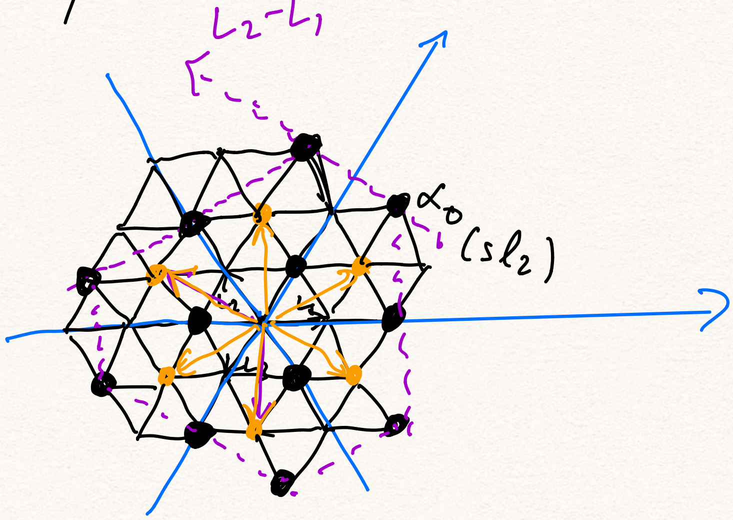
$= V_{(0,0)}$ trivial 1-d rep.

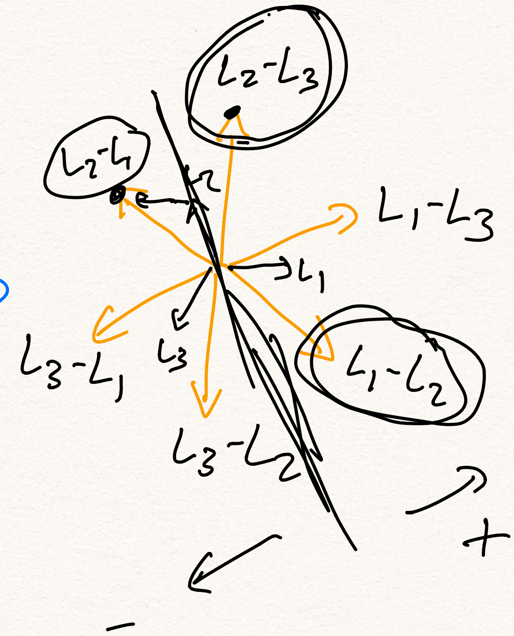
② 

$= V_{(1,0)}$ adjoint rep

3  $V_{(1,1)}$ Not the adjoint rep!?

$\mathfrak{sl}_3$-rep theory





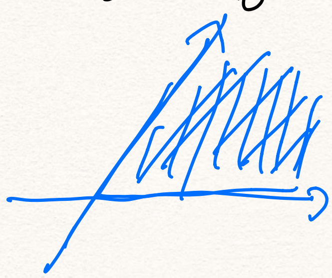
Irr. $\mathfrak{sl}_3$-reps look like:

$$\underset{\substack{= \\ \uparrow}}{V_{\alpha_0}} = \bigoplus_{\alpha \in S_{\alpha_0}} V_\alpha$$

highest weight

where S_{α_0} is the set of lattice pts forming a triangular grid (differing by roots) that fit inside the hexagon obtained by applying reflections to α_0 across L_1, L_2, L_3

Also: α_0 is in



Labeling α_0

$$\alpha = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3$$

$$= \alpha_1 L_1 + \alpha_2(-L_1 - L_3) + \alpha_3 L_3$$

(Recall: $L_1 + L_2 + L_3 = 0$)

$$= \underline{(\alpha_1 - \alpha_2)} L_1 - \underline{(\alpha_2 - \alpha_3)} L_3$$

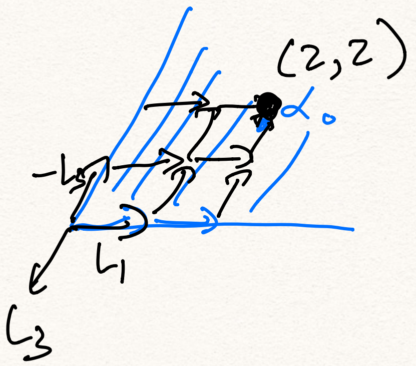
$$\alpha \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} = \alpha_1 - \alpha_2$$

$$\boxed{\alpha = a L_1 - b L_3}$$

Write this weight as (a, b)

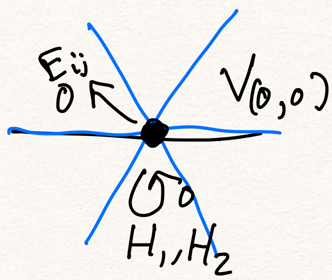
__Lemma:__ If $\alpha_0 = (a, b)$ is highest weight, $a, b \in \mathbb{N}$.

__Pf:__ by picture.

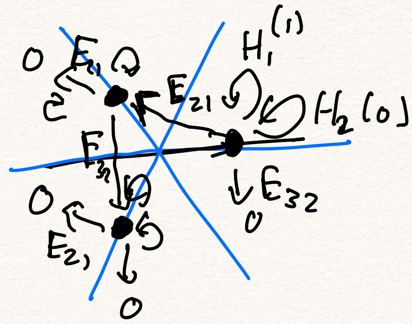


(from thm last time, α_0 is in blue shaded region).

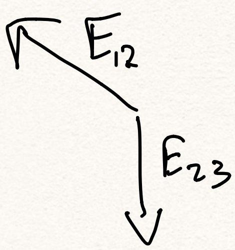
__Cor:__ One irred. rep $V_{(a,b)}$ for every pair $(a,b) \in \mathbb{N}^2$ (for sl_3)

__Ex:__ ① 

trivial rep $\mathbb{C} = V_{(0,0)}$
sl_3 acts by 0

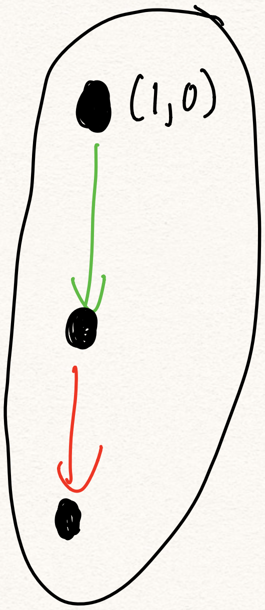
② __$V_{(1,0)}$:__ 

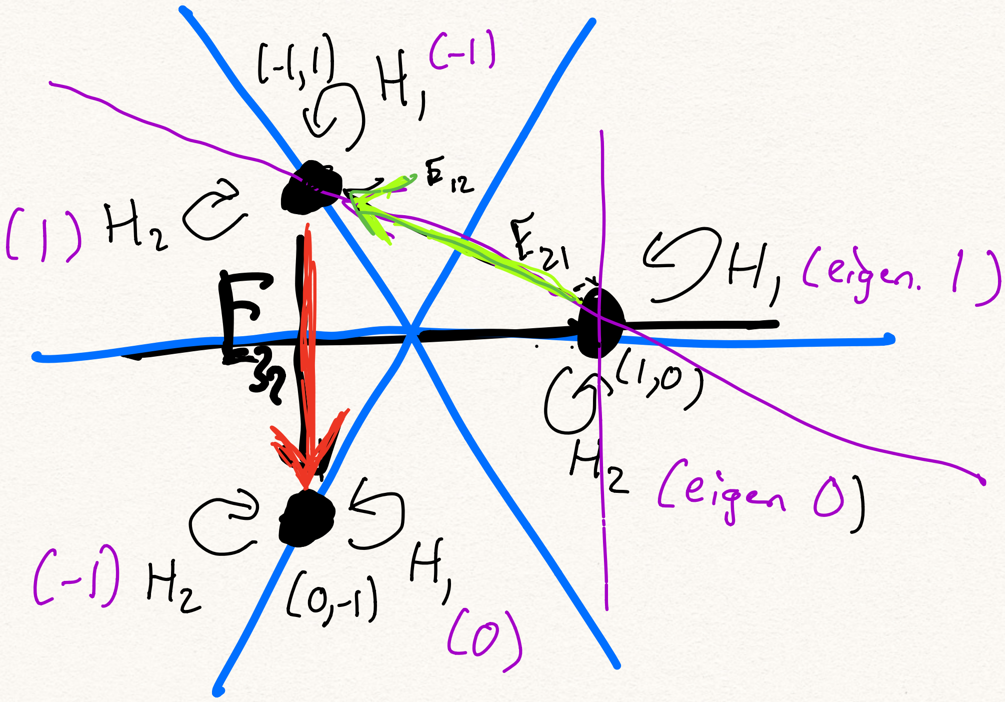
$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$H_1 = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix}$$

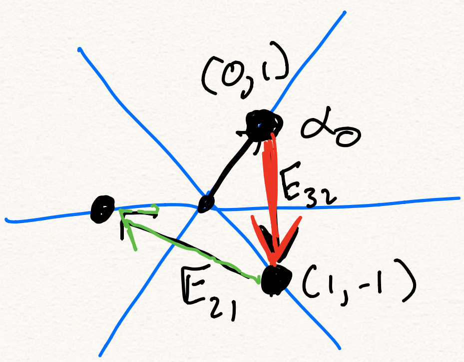
$$H_2 = \begin{pmatrix} 0 & 0 \\ 0 & \boxed{\begin{matrix} 1 & \\ & -1 \end{matrix}} \end{pmatrix}$$

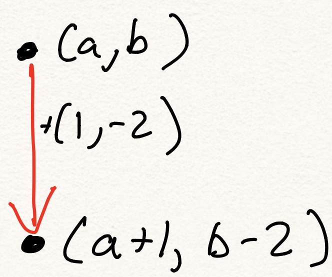


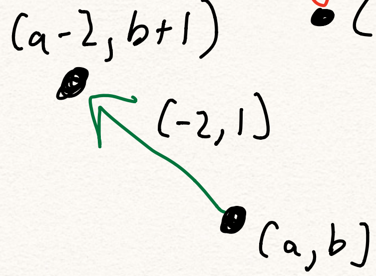


$$[E_{12}, E_{23}] = 2E_{13}$$

③ $V_{(0,1)}$:



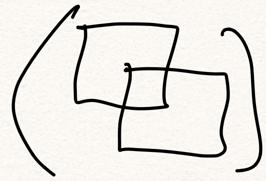


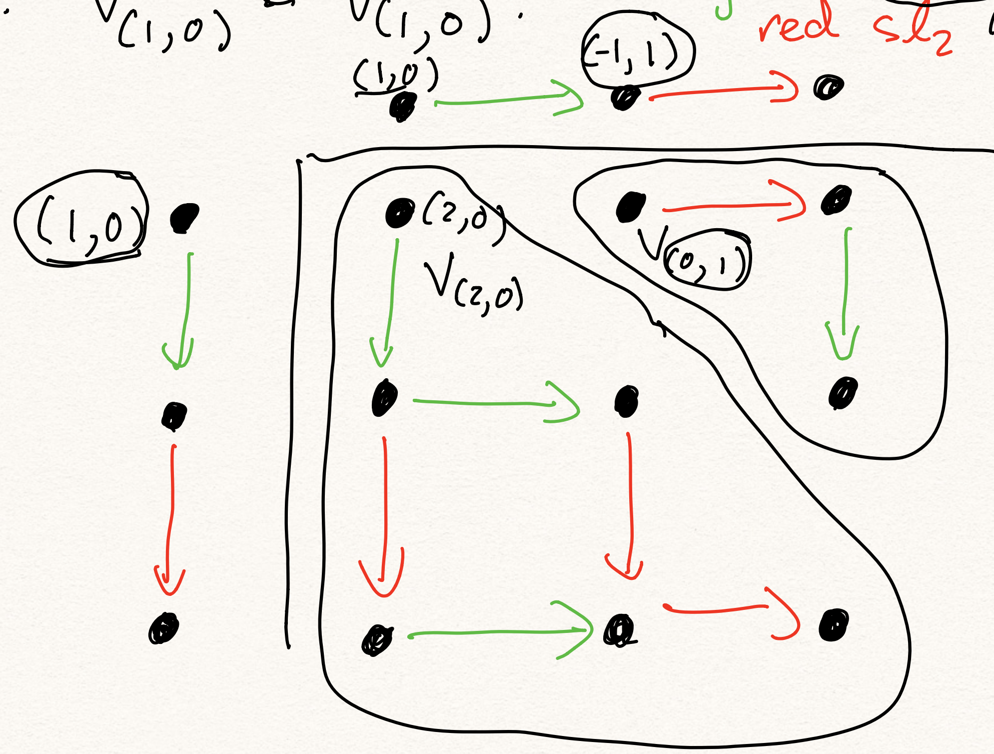


Tensor products:

Ex: $V_{(1,0)} \otimes V_{(1,0)}$:

green $\mathfrak{sl}_2 \subseteq \mathfrak{sl}_3$
red $\mathfrak{sl}_2 \subseteq$





Think of $V_{(1,0)}$ as an $\mathfrak{sl}_2$ rep.

$\mathfrak{sl}_2 \subseteq \mathfrak{sl}_3 \longrightarrow \mathfrak{gl}(V_{(1,0)})$

restrict: $\mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V_{(1,0)})$

"forget all red arrows"

$E_{12}, H_1; E_{21}$:

$$\left(\begin{array}{c|c} \boxed{\mathfrak{sl}_2} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$\mathfrak{sl}_3 \curvearrowright V_{(1,0)} \otimes V_{(1,0)}$

$g \cdot (v \otimes w) = gv \otimes w + v \otimes gw$

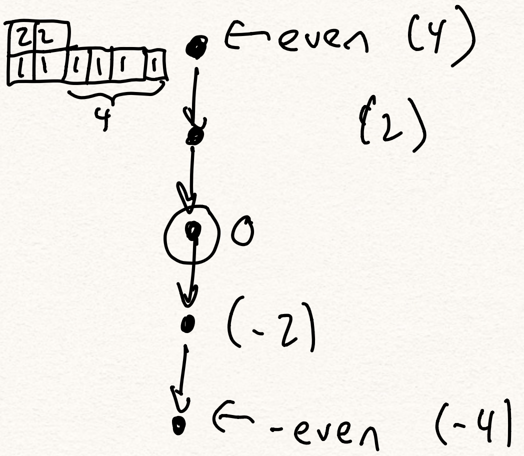
If $g \in \mathfrak{sl}_2$: $g(v\otimes w) = gv \otimes w + v \otimes gw$

Homework A problem 7(a):

Show # ballot sequences of 1's and 2's
of length $2n$ is $\binom{2n}{n}$ and
of length $2n+1$ is $\binom{2n+1}{n+1}$.

<u>Proof using sl_2 chains:</u> ($2n$ case first)

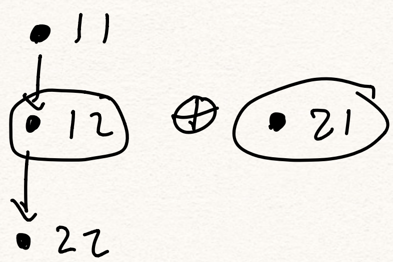
Ballot sequences correspond to highest wt elts of each sl_2 chain in $V_1^{\otimes 2n}$.



Every chain in $\boxed{V_1^{\otimes 2n}}$ has even highest weight
$\Rightarrow$ every chain has a unique weight-0 elt.
(corresponds to a word having n 1's and n 2's)

Moreover, every word having n 1's and n 2's is in a unique chain in $\underbrace{V_1^{\otimes 2n}}_{\{\text{all words in 1's, 2's}\}}$

$V_1^{\otimes 2}$:



words w/ n 1's and n 2's $= \binom{2n}{n}$

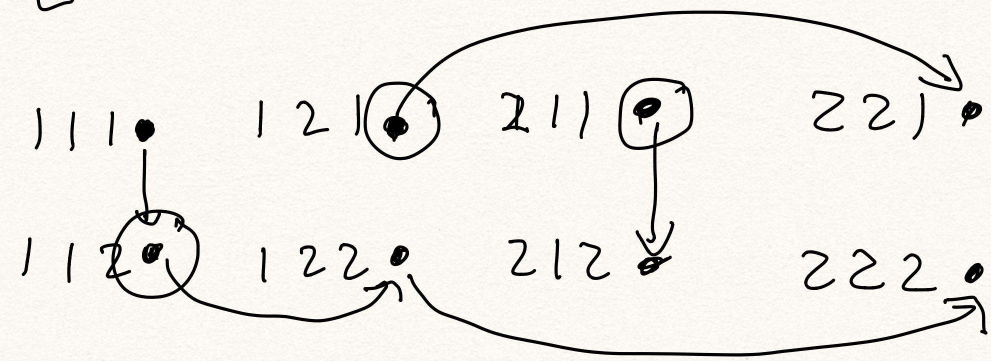
<u>Odd case:</u> all weights are odd:
$\Rightarrow$ Every chain has an elt of weight 1

$(n+1 \text{ 1's}, \; n \text{ 2's})$

$$\# = \binom{2n+1}{n+1}$$

<u>Ex:</u> V_1:

$V_1^{\otimes 3}$:

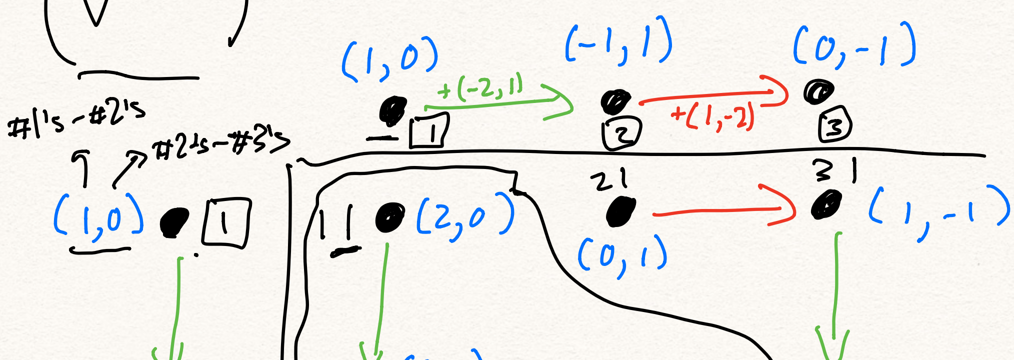


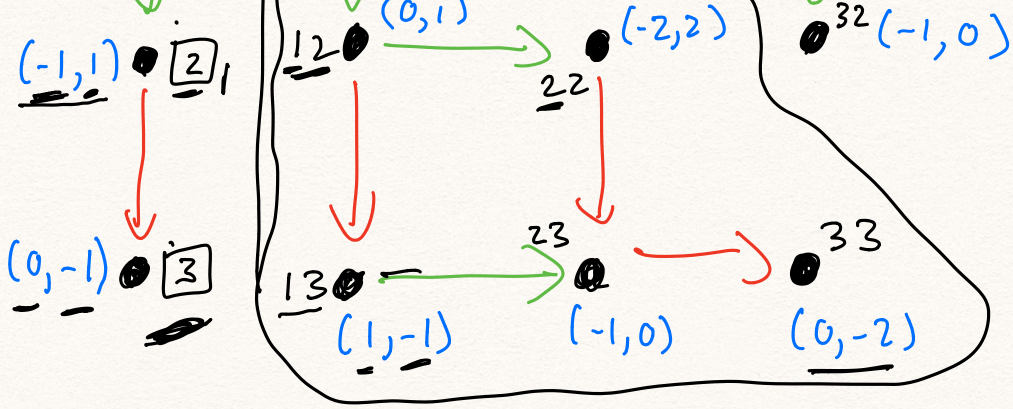
<u>Def:</u> $\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_n}$ (or simply $a_1 a_2 \cdots a_n$) denotes the weight space corresponding to $\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_n}$ in $\underline{V_{(1,0)}}^{\otimes n}$

<u>Thm:</u> (Goal): Every irred sl_3-rep $V^{(a,b)}$ is a summand of some $\left(V^{(1,0)}\right)^{\otimes n}$ for some n.

(i.e. we can just use word/tableau combinatorics)

$\underline{\left(V^{(1,0)}\right)^{\otimes 2}}$





Recall: sl_2 weight is eigenvalue of H.

sl_3 weight is eigenvalues (H_1, H_2)

$$H(v \otimes w) = Hv \otimes w + v \otimes Hw$$
$$= (\alpha + \beta)\, v \otimes w$$

(set $H = H_1$ or H_2, each eigenvalue adds)

Lemma: $\mathrm{wt}(a_1 a_2 \cdots a_n)$ is $(\#1\text{'s} - \#2\text{'s},\ \#2\text{'s} - \#3\text{'s})$

Pf: By induction, and additivity of weights. $\square$

Lemma: $F_1 := E_{21}$ applied to $a_1 \cdots a_n \in \{1,2,3\}^n$ is the word formed by bracketing 2's and 1's
( )
and changing rightmost unpaired 1 to 2

$F_2 := E_{32}$ applied to $a_1 \cdots a_n$ is formed by bracketing 3's w/ 2's and changing
( )
rightmost unpaired 2 to 3.

Pf: By induction. Base case:

$V^{(1,0)}$:

$$\begin{tikzcd} \bullet\ 2 \arrow[d, "F_2"'] \\ \bullet\ 3 \end{tikzcd}$$

<u>Induction step:</u> Assume true for $V_{(1,0)}^{\otimes(n-1)}$.

Tensor w/ $\underline{V_{(1,0)}}$ to add new letter $(1,2,3)$

Then F_1 structure given by tensoring $V_{(1,0)}^{\otimes(n-1)}$ as sl_2-module with

$$\begin{tikzcd} \boxed{1} \arrow[d] \\ \boxed{2} \end{tikzcd} \oplus \boxed{3}$$

Adding a 3 @ end of word is tensoring w/ trivial $\Rightarrow$ no change in how F_1 applies.

Adding 1's or 2's continue bracketing rule by our inductive pf in sl_2 case.

QED.

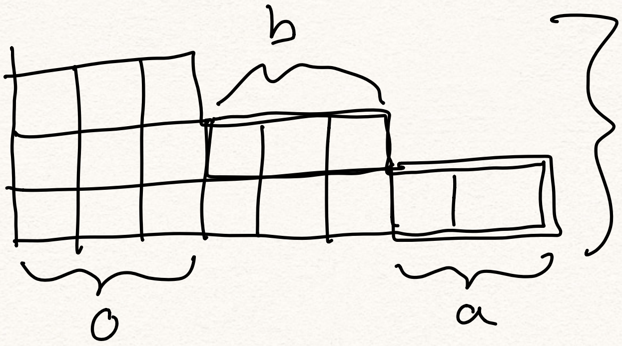
(F_2 similar) $\quad \boxed{1} \oplus \begin{tikzcd} \boxed{2} \arrow[d] \\ \boxed{3} \end{tikzcd}$

<u>Corollary:</u> A word is <u>highest weight</u> (killed by raising operators $E_1 = E_{12}$, $E_2 := E_{23}$) iff its 1,2-subword is ballot (every suffix contains at least as many 1's as 2's) <u>and</u> its 2,3-subword is ballot. (every suffix contains at least as many 2's as 3's)

<u>Note:</u> One irred. component for each highest

weight elt.

Thm: Number of times $\underline{V_{(a,b)}}$ occurs in $\underline{V_{(1,0)}^{\otimes n}}$ is # SYT's of shape:



Ex: $V_{(1,0)}^{\otimes 3}$:

111
$\downarrow F_1$
112

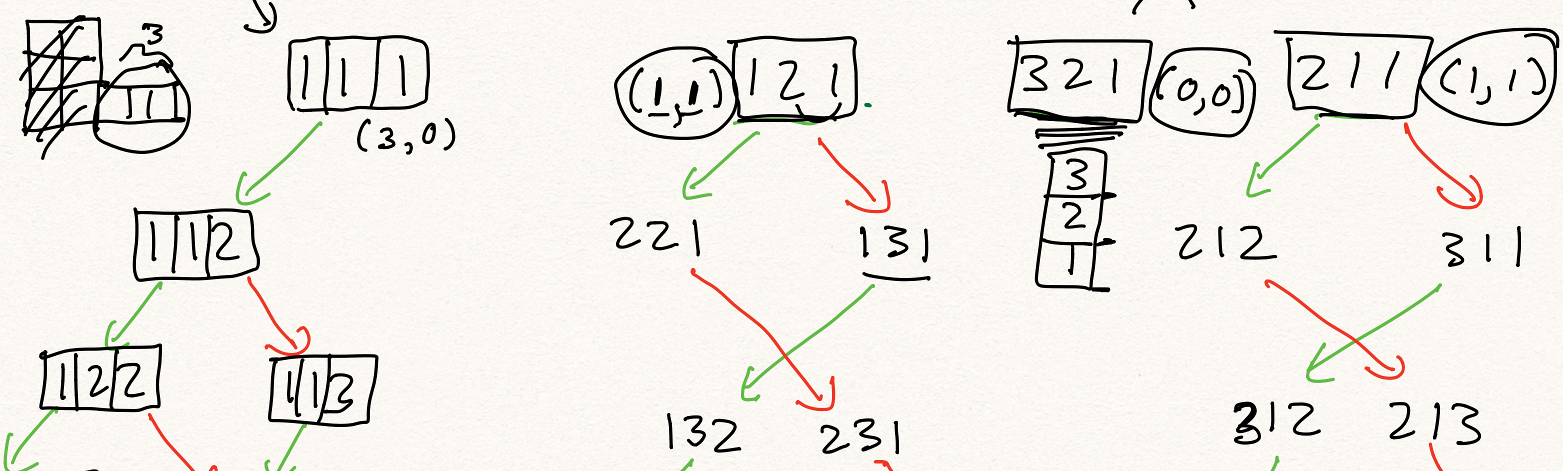
121

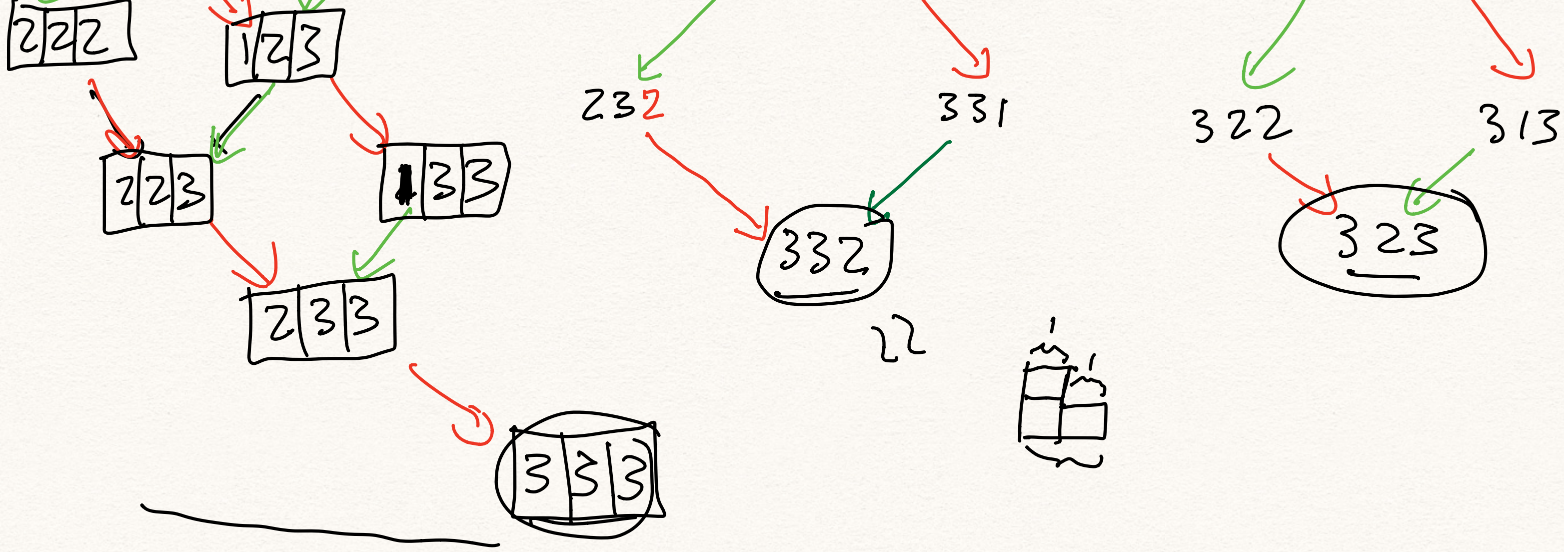
⋮

Shortcut:
Find all h.w. words first. (ballot: as read R to L, at least as many i's as i+1's)

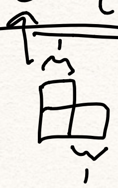
H.W. words: $\boxed{111,\ 121,\ 321,\ 211}$

Length 4 hw words: 1111, 2121, 3121,
3211, 1121, 1211, 2111, 2211
1321

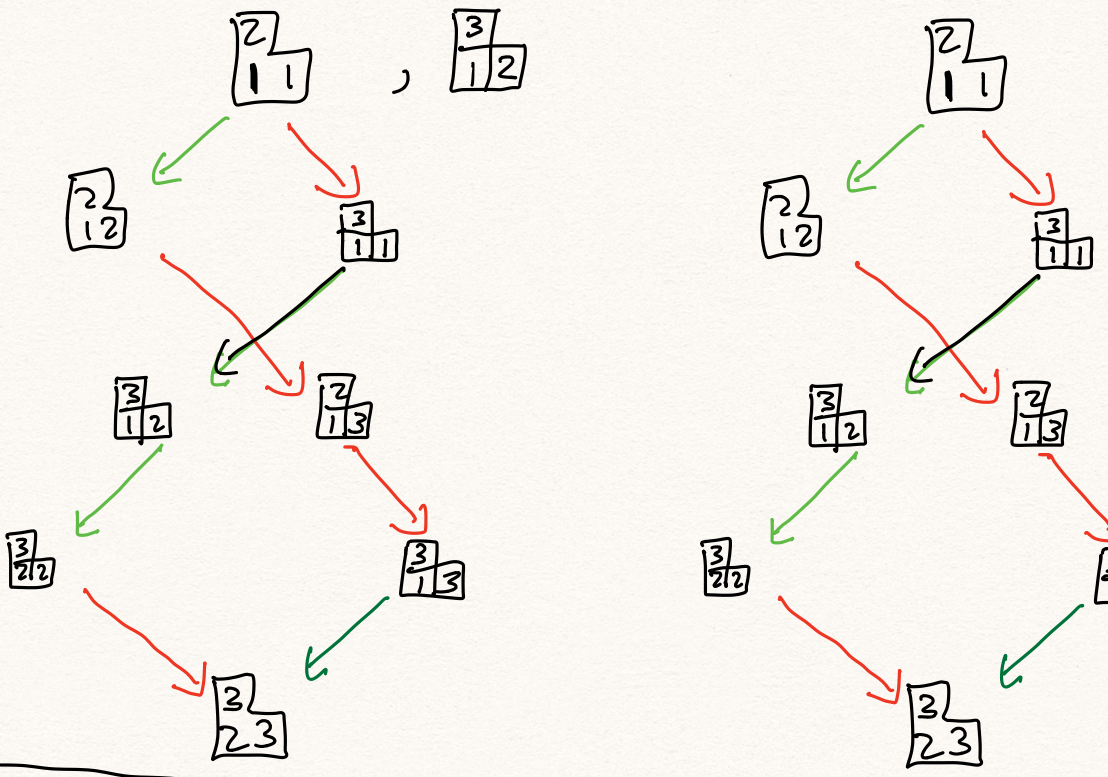


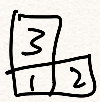


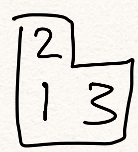
$$\boxed{V_{(1,0)}^{\otimes 3} = V_{(3,0)} \oplus 2V_{(1,1)} \oplus V_{(0,0)}.}$$



Apply RSK to these two diagrams: (words 121, 211)



Rec tab: 

Rec: 

$\Rightarrow$ # copies is # possible recording tableaux on this shape, which equals #SYT of this shape.

<u>Claim:</u> Insertion tableau of any highest weight word is

$$\begin{array}{|llllll} 3 & 3 & & & & \\ 2 & 2 & 2 & & & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

b, a, $\in$,

<u>Steps:</u> (1) Knuth equivalence classes $\Leftrightarrow$ insertion tableau.

[(2) Knuth equivalence moves don't change word of unbracketed 1's, 2's or 2's, 3's.

(3) Reading word of $\begin{array}{|lllll} 3 & 3 & & & \\ 2 & 2 & 2 & & \\ 1 & 1 & 1 & 1 & 1 \end{array}$ is highest weight (ballot) $\boxed{33222111111}$

(1) <u>Lemma:</u> Two words are Knuth equiv iff they have the same insertion tableau.

<u>Proof:</u> Recall a <u>simple Knuth move</u> is one of:

(cons. subwords) $\quad bac \Leftrightarrow bca \quad$ if $a < b \le c$

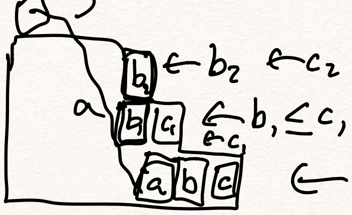
OR $\quad acb \Leftrightarrow cab \quad$ if $a \le b < c$

Two words w, w' are <u>Knuth equiv</u> if we can form w' from a sequence of Knuth moves starting at w.

($\Rightarrow$) Suppose w, w' differ by a Knuth move.

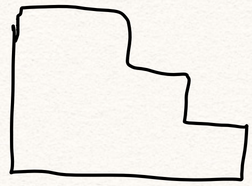
<u>Case 1:</u> $\underbrace{w_1 \cdots w_k\, bac \sim}_{w} \rightarrow \underbrace{w_1 \cdots w_k\, bca \sim}_{w'}$.

Inserting b:



$\leftarrow b \leftarrow a \leftarrow c$

$(\text{ins } w_1 \cdots w_n)$



$\leftarrow b \leftarrow c \leftarrow a$

$(\text{ins } w_1 \cdots w_n$

<u>Recall (502)</u>: Insertion path goes up and weakly left.

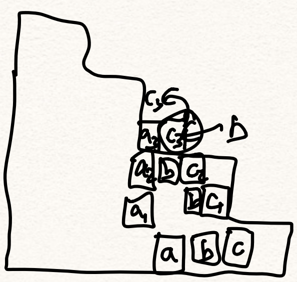
$a \le b$: Insertion path of a is weakly left of that of b.

$b < c$: Insertion path of c (after a) is <u>strictly</u> to right of b's path.

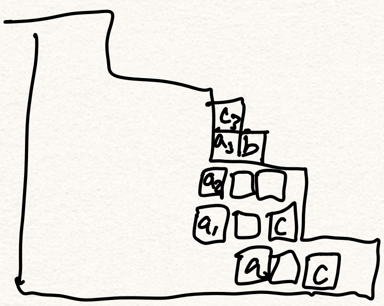
In w': c's path still strictly right of b's, a's is weakly left of b's

so $\text{ins}(w) = \text{ins}(w')$.

<u>Case 2:</u> $acb \leftrightarrow cab$



$\leftarrow a \leftarrow c \leftarrow b$



$\leftarrow c \leftarrow a$

<u>Insert b</u>: path is strictly right of a's, weakly left of c's.

($\Leftarrow$) Want to show: If $\underline{\text{ins}(w) = \text{ins}(v)} = \underline{T}$ then $\underline{w \sim v}$

Suffices to show they are both Knuth equivalent to the reading word of T

(Note: $\underline{\mathrm{ins}(\mathrm{rw}(T)) = T}$)
(not hard, possibly 502)

Want: if $\mathrm{ins}(w) = T$ then $w \sim \mathrm{rw}(T)$.

Claim: $\mathrm{rw}(T') \cdot x \sim \mathrm{rw}(T' \leftarrow x)$ (this suffices by inducting on length of w)

letter

concatenation

inserting x into T'

$w = w_1 w_2 \cdots w_n$ $\mathrm{ins}(w) = \left(\boxed{w_1} \leftarrow w_2\right) \leftarrow w_3 \leftarrow \ldots$

Pf of claim: (by example)

$$T' = \begin{array}{l} 3\,4 \\ 2\,2\,3 \\ 1\,1\,1\,2\,2\,3\,5 \end{array} \qquad x = 1$$

$T' \leftarrow x$:

$$\begin{array}{l} \square \\ 3\,\boxed{4} \\ 2\,2\,\boxed{3} \\ 1\,1\,1\,\boxed{2}\,2\,3\,5 \leftarrow 1 \end{array} \rightsquigarrow \begin{array}{l} 4 \\ 3\,3 \\ 2\,2\,2 \\ 1\,1\,1\,1\,2\,3\,5 \end{array} \qquad T = T' \leftarrow x$$

Compare:

$\mathrm{rw}(T') \cdot x$ vs $\mathrm{rw}(T)$

$= 3422311122351$ $4332221\underbrace{111235}$

2

3422311122315 4332221111235

2

3422311122135

2

$$3422311\,1\boxed{2}1\,235$$

$$\wr$$

$$342231\,121\,1235$$

$$\wr$$

$$34223\,121\,11235$$

$$\wr$$

$$34223\,2\,\underbrace{1111235}$$

$$\wr$$

$$34\,232\,21111235$$

$$\wr$$

$$343\,2221111235$$

$$\wr$$

$$4332221111235$$

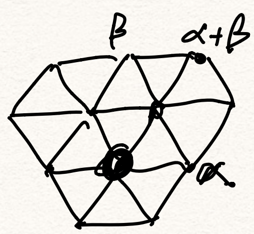
QED

Characters and Schur functions

Def: The character of a rep V of Lie alg $\mathfrak{g}$ (where $V = \underbrace{\bigoplus_{\alpha \in S} c_\alpha V_\alpha}_{\text{weight spaces}}$) is $\sum_{\alpha \in \Lambda} c_\alpha \underline{x^\alpha}$

where $\boxed{x^\alpha}$ is a formal symbol satisfying

$$\boxed{x^\alpha x^\beta = x^{\alpha+\beta}}$$



Ex: chars of sl_3 reps: $\underline{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}} =: x^\alpha$

where $\alpha = \underline{\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3}$

$$L_1 + L_2 + L_3 = 0$$

Polynomials in x_1, x_2, x_3 mod relation

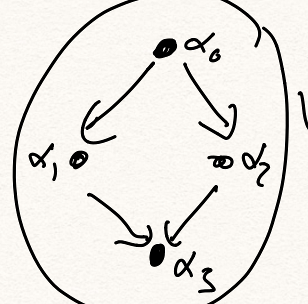
$$\boxed{x_1 x_2 x_3 = 1}$$

$$x_1^4 x_2^2 x_3 = x_1^3 x_2^1$$

$V^{(a,b)}$: irr. rep. w/ highest wt (a,b)

V_α : weight space for α in rep V.

- $V^{(a,b)}_{(a,b)}$



$$\underline{V^{\alpha_0} = V_{\alpha_0} \oplus V_{\alpha_1} \oplus V_{\alpha_2} \oplus V_{\alpha_3}}$$

__Ex:__

$$\begin{tikzcd}
 & \boxed{1\,|\,1} \arrow[dl, "F_1"'] & \\
12 \arrow[d, "F_1"'] \arrow[dr, "F_2"] & & \\
22 \arrow[dr, "F_2"'] & 13 \arrow[d] & \\
 & 23 \arrow[d] & \\
 & 33 &
\end{tikzcd}$$

<u>char.</u>

$x_1^2 \qquad x_1^2 x_2^0 x_3^0$

$x_1 x_2$

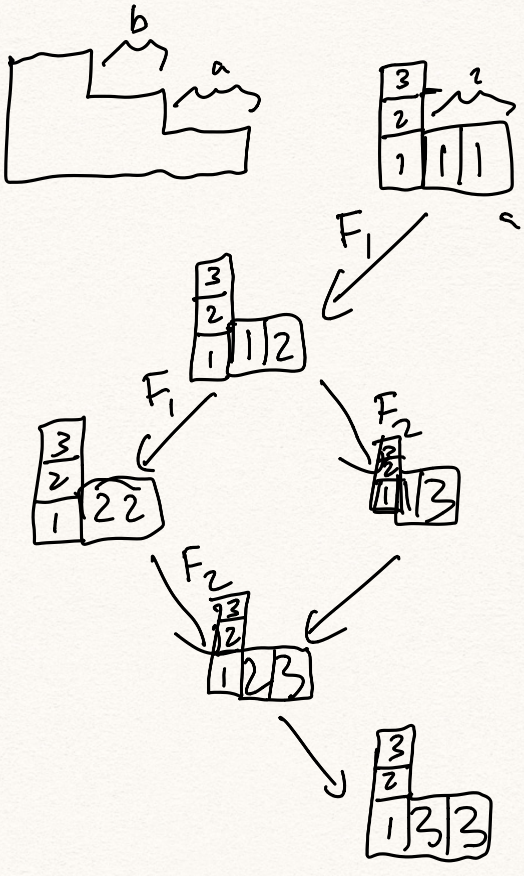
$x_2^2, \ x_1 x_3$

$x_2 x_3$

x_3^2

$$\underbrace{\qquad\qquad}_{x_1^{\#1\text{'s}} \, x_2^{\#2\text{'s}} \, x_3^{\#3\text{'s}}}.$$

$$\rightsquigarrow \operatorname{ch}(V^{(2,0)}) = x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2 = s_{(2,0)}(x_1, x_2, x_3)$$



$V^{(2,0)}$

<u>char:</u>

$x_1^3 x_2 x_3$

$+$

$x_1^2 x_2^2 x_3$

$+$

$\vdots$

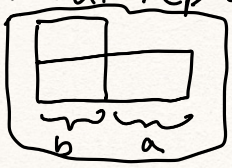
$$s_{(3,1,1)}(x_1,x_2,x_3) = \underbrace{x_1x_2x_3}\left(s_{(2,0)}(x_1,x_2,x_3)\right)$$

α's are joint eigenvalues of $\mathfrak{h}$ = diag. matrices in $\mathfrak{g}$.

(trace = sum of eigenvalues)

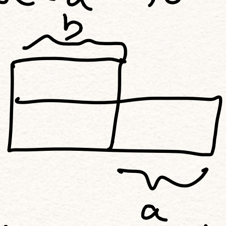
<u>Lemma:</u> $\mathrm{ch}(V^{(a,b)}) \equiv s_{(a,b)}(x_1,x_2,x_3)$ (mod $x_1x_2x_3=1$)

$\uparrow$ irred. rep of h.w. (a,b)

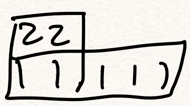


<u>Pf:</u> Recall
$$s_\lambda(x_1,\dots,x_n) = \sum_{\substack{\text{SSYT } T \\ \text{of shape } \lambda \\ \text{using only} \\ \text{letters } 1,\dots,n}} x_1^{\#1\text{'s}}\, x_2^{\#2\text{'s}} \cdots x_n^{\#n\text{'s}}$$

Need to show <u>every</u> SSYT of shape



w/ 1's, 2's, 3's can be obtained by sequence of F_1's, F_2's applied to



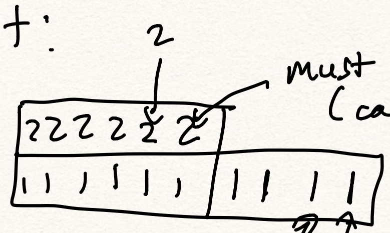
Given an SSYT T, if it's not h.w, we can apply E_1 or E_2 until get to a highest weight tab.

E_1: Bracket 1's, 2's, change 2 to 1

E_2: Bracket 2's, 3's, change 3 to 2.

get to a tab. S whose reading word

is ballot:

$S =$ 

must be 2 (can't be 3 by ballotness)

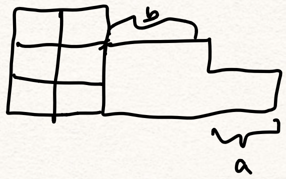
must be 1 by ballotness

must be 1 by semist.

Reverse this process to get from S to T w/ F_1, F_2. ✓

(Today: Finishing sl_3, moving to sl_n)

Recall: $ch(V^{(a,b)}) = s_{\underbrace{(a+b,b)}_{\text{partition}}}(\underline{x_1, x_2, x_3})$ (last time)



$(\text{mod } x_1 x_2 x_3 = 1)$

Lemma: ① $ch(V \otimes W) = ch(V) \cdot ch(W)$ for any reps V, W of $\mathfrak{g}$

② $ch(V \oplus W) = ch(V) + ch(W)$

Pf: ① $ch(V \otimes W) = ch((\bigoplus_\alpha V_\alpha) \otimes (\bigoplus_\beta W_\beta))$ (weight spaces V_α, W_β: eigenspaces for $\mathfrak{h}$)

$$= ch(\bigoplus_{\alpha,\beta} (\underbrace{V_\alpha \otimes W_\beta}_{\substack{\text{weight spaces} \\ \text{for } V \otimes W \text{ of weight} \\ \alpha+\beta.}}))$$

$$= \sum_{\alpha,\beta} x^{\alpha+\beta}$$

In sl_3: $(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3})$

$$= \sum_{\alpha,\beta} x^\alpha x^\beta$$

$$= \left(\sum_\alpha x^\alpha\right)\left(\sum_\beta x^\beta\right)$$

$$= ch(\oplus V_\alpha) \cdot ch(\oplus W_\beta)$$

$$= ch(V) \cdot ch(W).$$

② $ch(V \oplus W) = ch(\bigoplus_\alpha V_\alpha \bigoplus_\beta W_\beta)$

$$= \sum_\alpha x^\alpha + \sum_\beta x^\beta$$

$= ch(V) + ch(W)$. QED.

Cor (sl_3 case):

① Every character of an sl_3-rep $ch(V) = ch\left(\bigoplus_{(a,b)\in S} V^{(a,b)}\right)$ is a positive sum of Schur functions $s_\lambda(x_1, x_2, x_3)$ (where λ has at most 2 parts)

i.e. $$ch(V) = \sum_{\substack{\lambda \text{ part.} \\ \text{w/ at most 2 parts}}} c_\lambda s_\lambda$$ (schur positive)

where $c_\lambda \in \mathbb{N}$.

② If $ch(V) = \sum c_\lambda s_\lambda(x_1, x_2, x_3)$

then $V = \bigoplus_\lambda c_\lambda V^\lambda$

③ V^ν appears in $V^\lambda \otimes V^\mu$ exactly $c_{\lambda\mu}^\nu$ times where $c_{\lambda\mu}^\nu$ is coeff. of s_ν in $s_\lambda s_\mu$.

④ $c_{\lambda\mu}^\nu$ = # pairs of SSYT of shapes (in 1,2,3) whose concatenated reading word is ballot of content ν.

(Pf: $\underbrace{V^\lambda}_{\substack{\text{wt spaces:} \\ \text{SSYT of shape} \\ \lambda\ (1,2,3)}} \otimes \underbrace{V^\mu}_{\substack{\text{SSYT} \\ \text{shape } \mu \\ (1,2,3)}}$: highest wt elts?

$\times (1,2,3)$ $\qquad\qquad$ $(1,2,3)$

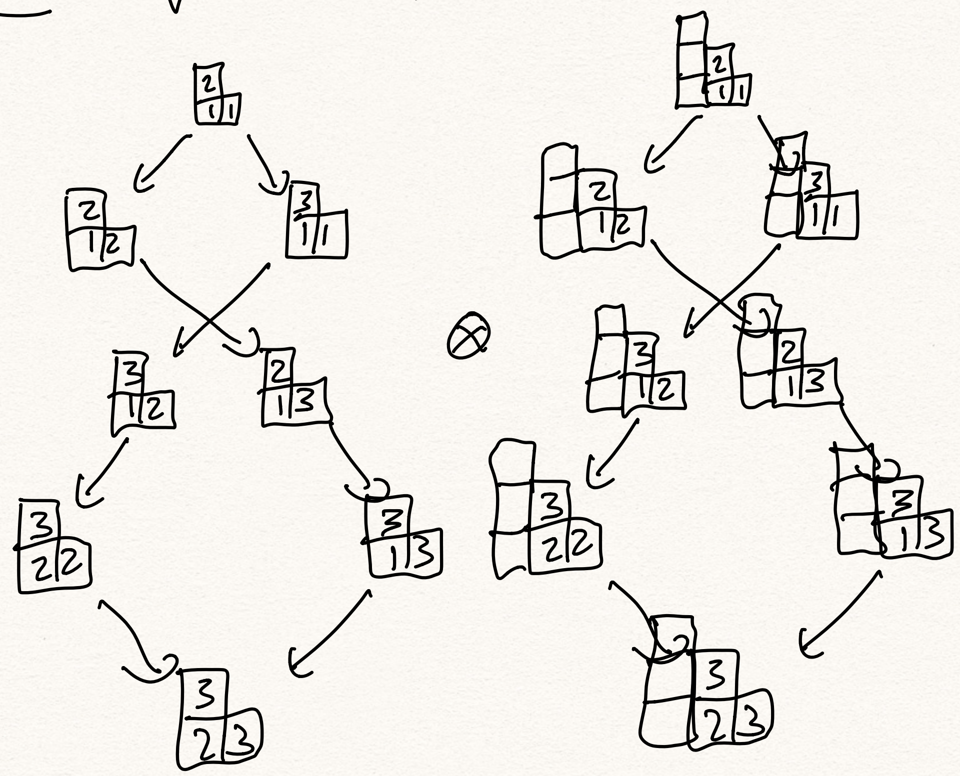
$\Updownarrow$ $\qquad\qquad\qquad$ $\Updownarrow$

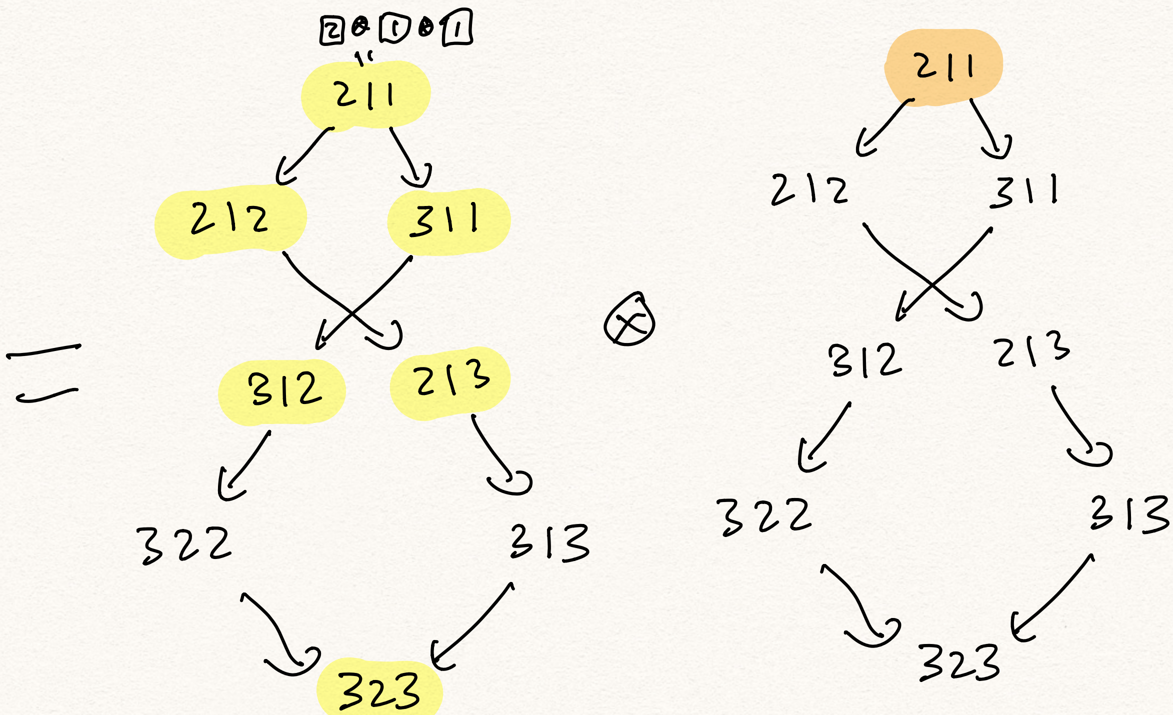
reading word $\subseteq (V^{(1,0)})^{\otimes n}$ $\qquad$ reading word $(V^{(1,0)})^{\otimes n}$

Hw. iff concatenated word is ballot.)

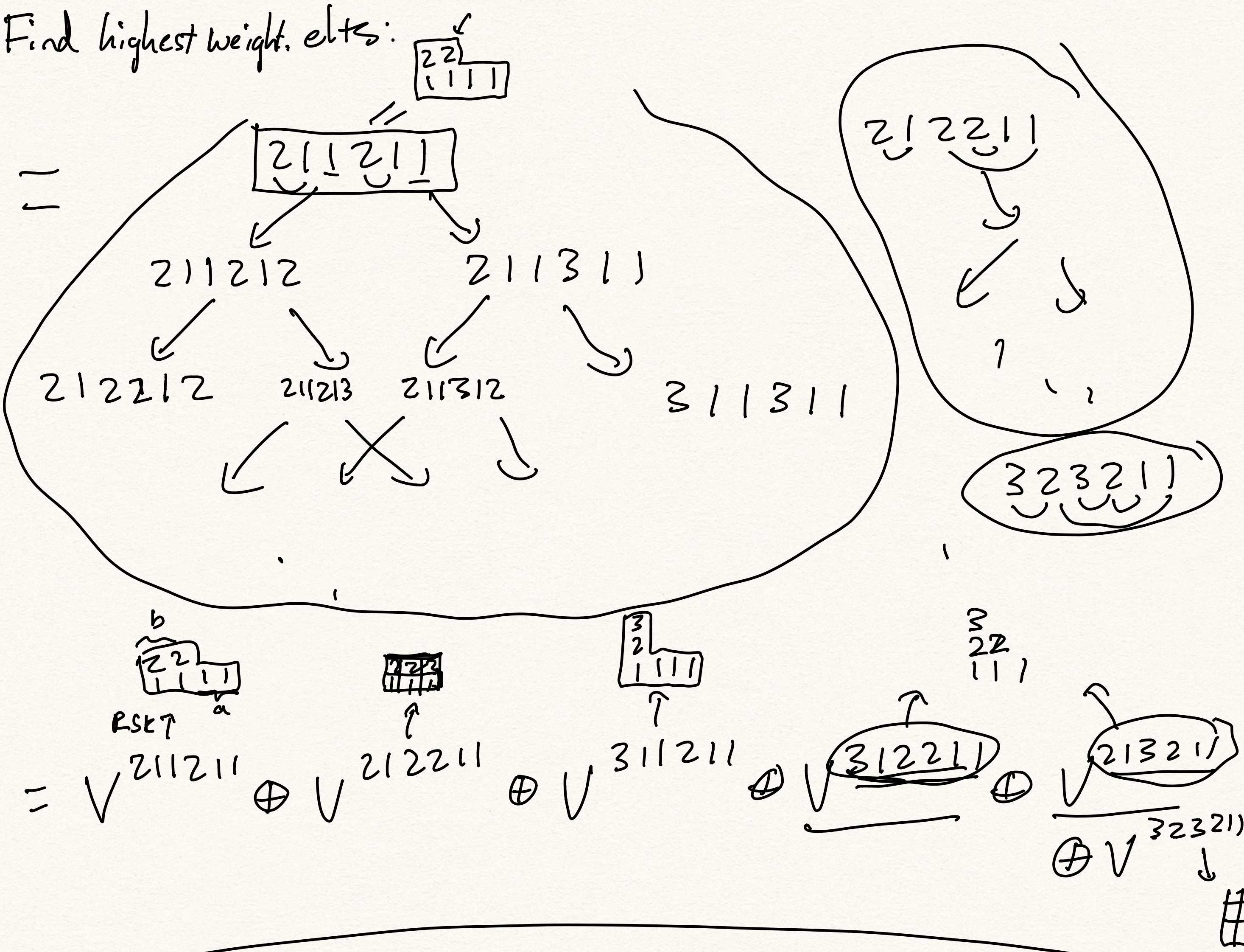
Ex: $V^{\square} \otimes V^{\square}$:

211 ~ 121





Find highest weight. elts:



$= V^{211211} \oplus V^{212211} \oplus V^{311211} \oplus V^{312211} \oplus V^{213211} \oplus V^{323211}$

RSK

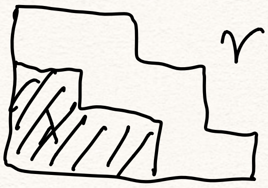
$$\boxed{= V^{(2,2)} \oplus V^{(0,3)} \oplus V^{(3,0)} \oplus 2V^{(1,1)} \oplus V^{(0,0)}}$$

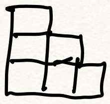
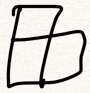
$$s_{(2,1)} \cdot s_{(2,1)} = s_{(4,2)} + s_{(3,3)} + s_{(4,1,1)} + 2\, s_{(3,2,1)} + s_{(2,2,2)}$$

$c^{(3,2,1)}_{(2,1),(2,1)} = 2 = \#$ pairs of SSYT's on shapes $((2,1),(2,1))$ that concatenate to ballot.

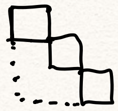
Littlewood-Richardson rule: Combinatorial formula for $c^{\nu}_{\lambda\mu}$

Alt. LR rule: $c^{\nu}_{\lambda\mu} = \#$ SSYT of shape ν/λ, content μ, Ballot reading word.

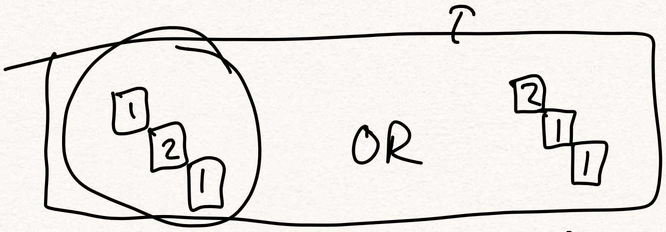


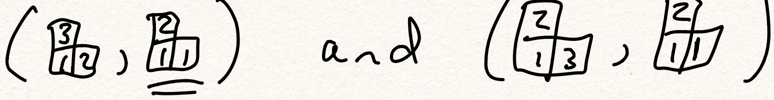
Ex: $\nu =$  , $\lambda = \mu =$ 

content μ : content (2)(1) (two 1's, one 2)

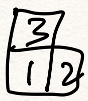
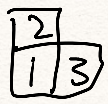
$\nu/\lambda =$ 

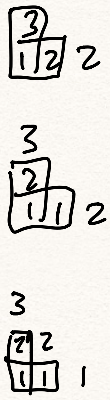
?



Bijection with  :

RSK insert 211 into either

 $\leftarrow$ 121 vs  $\leftarrow$ 121



(Homework)

Algebraic proof: $\boxed{\langle s_\nu, s_\lambda s_\mu \rangle = \langle s_{\nu/\lambda}, s_\mu \rangle}$

$$\langle s_\nu, s_\lambda s_\mu \rangle = c^\nu_{\lambda\mu}, \qquad \langle s_{\nu/\lambda}, s_\mu \rangle = c^\nu_{\lambda\mu}$$

Skew Schur function:

$$s_{(3,2,1)/(2,1)} = 2\,s_{(2,1)} + s_{(1,1,1)} + s_{(3)}$$