$$V_1\otimes V_1\otimes V_1 = V_1\otimes(V_2\oplus V_0)$$
$$= (V_1\otimes V_2)\oplus(\underbrace{V_1\otimes V_0})$$
$$= V_3\oplus V_1\oplus V_1$$

Recall: $sl_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix} : a,b,c\in\mathbb{C}\right\}$

$$= \{X\in Mat_2(\mathbb{C}) : tr(X)=0\}$$

Irred reps:

$$\begin{array}{cl} & \text{weights:} \\ \bullet & n \\ \downarrow & \\ \bullet & n-2 \\ \downarrow & \vdots \\ \bullet & \vdots \\ \downarrow & \\ \bullet & -n \end{array}$$

weights: eigenvalues of H

Schematic: in $V_1^{\otimes n}$

V_1: $\boxed{1} \leftarrow$, $\downarrow$, $\boxed{2} \leftarrow$

$V_1\otimes V_1$:

$$\begin{array}{c} \boxed{1}\otimes\boxed{1} \\ \downarrow F \\ \boxed{1}\otimes\boxed{2} + \boxed{2}\otimes\boxed{1} \\ \downarrow F \\ \boxed{2}\otimes\boxed{2} \end{array} \qquad \boxed{2}\otimes\boxed{1} - \boxed{1}\otimes 2$$

"quantum sl_2": $U_q(sl_2)$

gives q-analog of rep ~~theory~~ of sl_2

$U_q(sl_2)$-reps $V_1 \otimes V_1$:

$$\begin{array}{c} \boxed{1}\otimes\boxed{1} \\ \downarrow F \\ \boxed{1}\otimes\boxed{2} + \underline{q}\,\boxed{2}\otimes\boxed{1} \\ \downarrow F \\ \boxed{2}\otimes\boxed{2} \end{array} \qquad \boxed{2}\otimes\boxed{1} - \underline{q}\,\boxed{1}\otimes 2$$

$q=1$: get usual diagrams.

$q=0$: get $q \to 0$ limit (getting really cold)

"crystal"

$$\begin{array}{c} \boxed{1}\otimes\boxed{1} \\ \downarrow F \\ \boxed{1}\otimes\boxed{2} \\ \downarrow F \\ \boxed{2}\otimes\boxed{2} \end{array} \qquad \boxed{2}\otimes\boxed{1}$$

Review bracketing rule:

Ex: $V_1 \otimes V_1 \otimes V_1$

F arrows:

$$\begin{array}{cccc} 111 & 121 & 211 & 221 \\ \downarrow & & \downarrow & \\ 112 & 122 & 212 & 222 \end{array}$$

$121 \xrightarrow{F} 221$, $221 \to 0$, $112 \to 122$, $122 \to 222$, $212 \to 0$, $222 \to 0$

F: changes a 1 to a 2 (if possible)

1 2 2

) ((

2 2 1 1 1 2 1 1 1 2 2 1

(( ))) ( ))) (( )

E ↑↓ F

2 2 1 1 1 2 1 1 2 2 2 1

E ↑ ↓ F

↓ F

↓

0

$\boxed{1|2|3}$ $V_3 \oplus V_1 \oplus V_1 = V_1^{\otimes 3}$

V_3	V_1	V_1
111		
$\downarrow$	121	211
112	$\downarrow$	$\downarrow$
$\downarrow$	221	212
122	$\begin{array}{l}3\\1\,2\end{array}$	$\begin{array}{l}2\\1\,3\end{array}$
$\downarrow$		
222		

Ballot:
read right to left,
never more 2's than 1's

~~1 2 2 1~~

Q: How do we know when a word in 1's, 2's is at the top of a chain (highest weight)

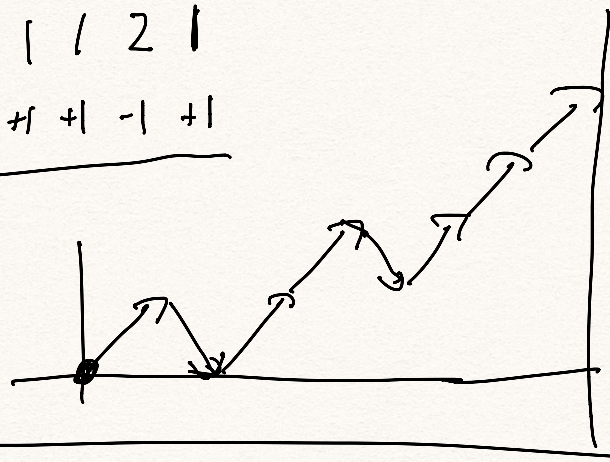
(Homework!)

Ballot: for any suffix, #1's $\geq$ #2's

1 1 1 2 1 1 2 1

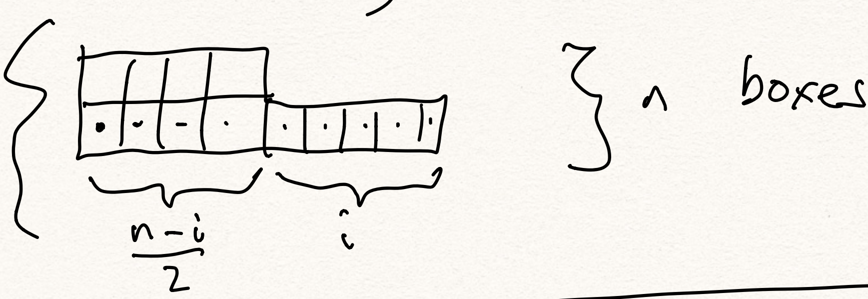
+1 +1 +1 −1 +1 +1 −1 +1

$\longleftarrow$



Q: How many times does $\underline{V_i}$ appear in $V_1^{\otimes n}$?

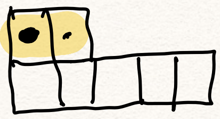
A: # standard Young tableaux of shape



$\lambda = \left(\frac{n+i}{2}, \frac{n-i}{2}\right)$ is the partition.

<u>Hook length formula:</u> # SYT of shape λ is

$$\frac{n!}{\prod_{s \in D(\lambda)} \text{hook}(s)}$$

<u>Ex:</u> $\lambda =$ 

$$\frac{7!}{6\cdot 5\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1} = \frac{7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}^{2}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{6}\cdot 5\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{2}} = 14$$

If $\lambda = \left(\frac{n+i}{2}, \frac{n-i}{2}\right)$:

$$\frac{n!\,(i+1)}{\left(\frac{n+i}{2}+1\right)!\left(\frac{n-i}{2}\right)!} = \boxed{\frac{(i+1)}{\left(\frac{n+i}{2}+1\right)}\binom{n}{\frac{n-i}{2}}}$$

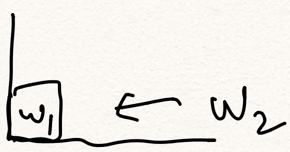
Why? <u>RSK</u> (Robinson-Schensted-Knuth)

<u>Thm</u> (Robinson): The following is a bijection

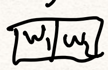
$$S_n \longrightarrow \{(S,T) : S, T \text{ are SYT's of the same shape } \lambda \vdash n\}$$

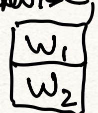
Given permutation $w = w_1 \cdots w_n$, <u>insert</u> letters one at a time into a corner

① 

② 

(new letter goes in bottom row)

If $w_2 > w_1$, put w_2 at end: 

Otherwise, bump w_1 up: 

③ $\begin{array}{|c|}\hline w_1 \\ \hline w_2 \\ \hline \end{array} \leftarrow w_3$

⋮

On step ⓘ: Insert w_i into tab T from step $i-1$

by: Case 1: If w_i larger than everything in bottom row of T, put it @ end.

Case 2: If not, bump out smallest # x bigger than w_i in bottom row, insert x into row 2, repeat.

Ex: 4 2 5 1 3 6

	Insertion	Recording
①	$\begin{array}{\|c\|}\hline 4 \\ \hline \end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline \end{array}$
②	$\begin{array}{\|c\|}\hline 4 \\ \hline 2 \\ \hline \end{array}$	$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline \end{array}$
③	$\begin{array}{ll} 4 & \\ 2 & 5 \end{array}$	$\begin{array}{ll} 2 & \\ 1 & 3 \end{array}$
④	$\begin{array}{ll} 4 & \\ 2 & \\ 1 & 5 \end{array}$	$\begin{array}{ll} 4 & \\ 2 & \\ 1 & 3 \end{array}$
⑤	$\begin{array}{ll} 4 & \\ 2 & 5 \\ 1 & 3 \end{array}$	$\begin{array}{ll} 4 & \\ 2 & 5 \\ 1 & 3 \end{array}$
⑥	$\begin{array}{lll} 4 & & \\ 2 & 5 & \\ 1 & 3 & 6 \end{array}$	$\begin{array}{lll} 4 & & \\ 2 & 5 & \\ 1 & 3 & 6 \end{array}$

Robinson: 425136 $\to \left(\begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 3 & 6 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 3 & 6 \\ \hline \end{array} \right)$

<u>Robinson-Schensted</u>: Bijection

$$W(\alpha_1, \alpha_2, \alpha_3 \dots) \longrightarrow \{(S,T) : S \text{ semistandard content } \alpha,\ T \text{ is SYT of same shape}$$

$$\| $$

$$\left\{ \text{words w/ } \begin{array}{l} \alpha_1 = \#1\text{'s} \\ \alpha_2 = \#2\text{'s} \\ \vdots \end{array} \right\}$$

<u>Ex</u>: 4251<u>1</u>6

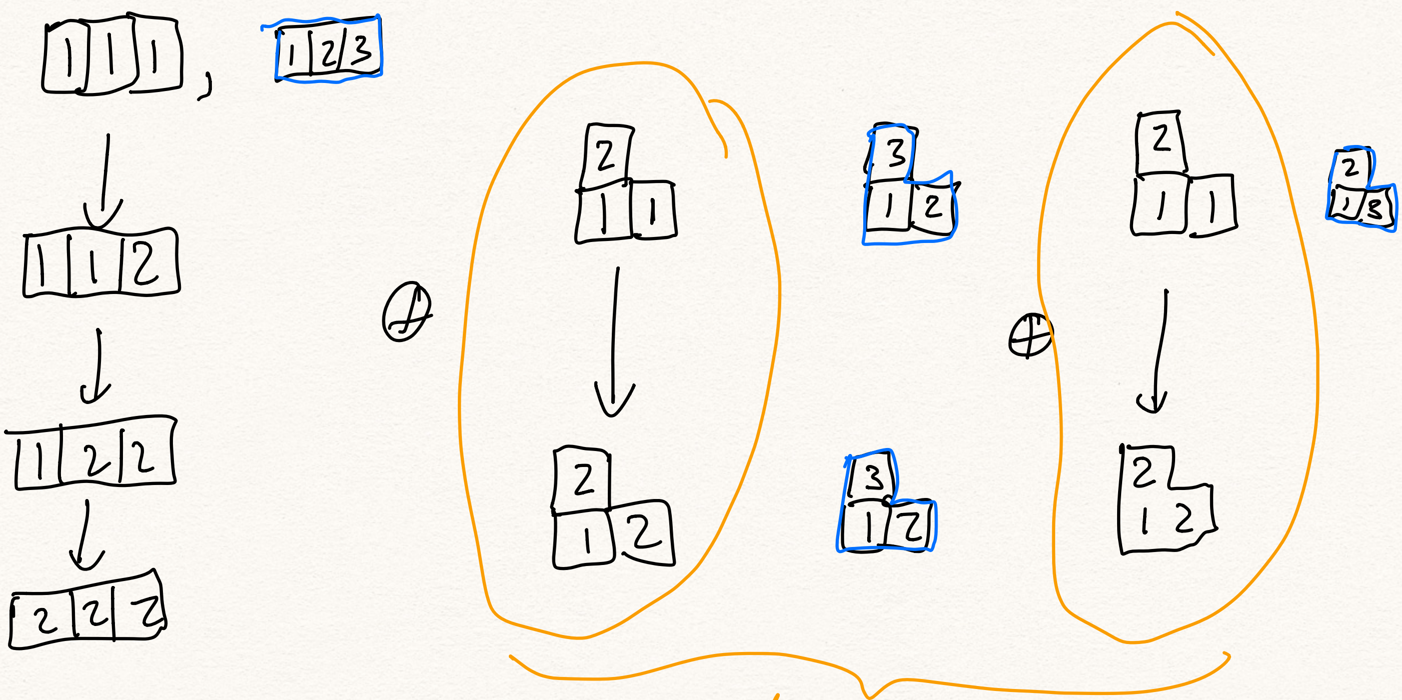
<u>Insertion</u>		<u>Recording</u>
1	$\begin{array}{|c|} \hline 4 \\ \hline \end{array}$	$\begin{array}{|c|} \hline 1 \\ \hline \end{array}$
2	$\begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline \end{array}$	$\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}$
3	$\begin{array}{|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 \\ \hline \end{array}$	$\begin{array}{|c|c|} \hline 2 \\ \cline{1-1} 1 & 3 \\ \hline \end{array}$
4	$\begin{array}{|c|c|} \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} 1 & 5 \\ \hline \end{array} \leftarrow 1$	$\begin{array}{|c|c|} \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} 1 & 3 \\ \hline \end{array}$
5	$\begin{array}{|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 1 \\ \hline \end{array}$	$\begin{array}{|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 3 \\ \hline \end{array}$
6	$\begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 1 & 6 \\ \hline \end{array}$	$\begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 3 & 6 \\ \hline \end{array}$

$\uparrow$

SSYT

RSK: apply to $V_1^{\otimes 3}$

$$111 \to 112 \to 122 \to 222 \quad \oplus \quad 121 \to 221 \quad \oplus \quad 211 \to 212$$



\# V_1 chains is \# tableaux shape ⊞.

<u>Lemma:</u> Let w be a word in alph. $\{1,2\}$. Consider bracketing of 2's w/ 1's. Let i = \# unbracketed letters. Then RSK ins. tab of w is

$$\begin{array}{|c|c|c|c|}\hline 2&2&2&2\\ \hline 1&1&1&1&1\,1\,1\,2\,2\,2 \\ \hline\end{array}$$
$$\underbrace{\qquad\qquad}_{i}$$

shape $\lambda = \left(\frac{n+i}{2}, \frac{n-i}{2}\right)$

where last i entries in bottom row are unbracketed #s in order.

Ex: $1\,1\,2\,2\,1\,2\,1\,1\;\underline{1\,1\,2\,2}$ $\to$ Fw

(guess:
$$\begin{array}{|c|c|c|}\hline 2&2&2\\ \hline 1&1&1&1\,1\,1\,1\,2\,2\\ \hline\end{array}$$

RSK:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 2&2&2\\ \hline 1&1&1&1&1&1&1&2&2\\ \hline\end{array}$$

rw: 2 2 2 1 1 1 1 1 1 1 2 2

$\Big\downarrow F$ $\qquad$ $\Big\downarrow F$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 2&2&2\\ \hline 1&1&1&1&1&1&2&2&2\\ \hline\end{array}$$

2 2 2 1 1 1 1 1 1 2 2 2

Lemma: If $w \xrightarrow{F} w'$ then RSK ins. tabs. of w, w' have same shape, and ins(w') obtained from ins(w) by changing last unpaired 1 to 2 in their reading words.

Moreover, recording tabs of w, w' are same:

RSK rec tab:

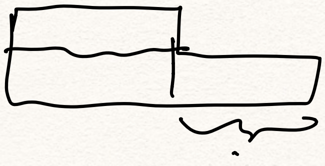
$w = \underline{1\,1}\;2\,2\,1\,2\,1\,1\;1\,1\,2\,2$

$$\begin{array}{|ccc|cccccc}\hline 5&6&7\\ \hline 1&2&3&4&6&8&9&10&11\\ \hline\end{array}$$

$w' = Fw = 1\,1\,2\,2\,1\,2\,1\,1\;1\,2\,2\,2$

$$\begin{array}{|ccc|cccccc}\hline 5&6&7\\ \hline 1&2&3&4&6&8&9&10&11\\ \hline\end{array}$$

Conclusion: There is one V_1 chain for every "recording tableau" of shape

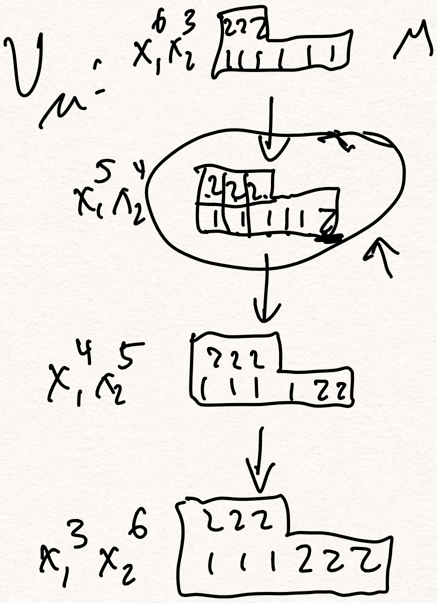


in $V_1^{\otimes n}$.

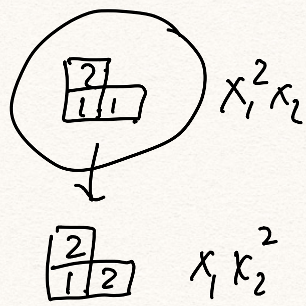
QED.

Q: What is $V_\mu \otimes V_\lambda$? (μ, λ both two-row partitions)

V_μ:



V_λ:

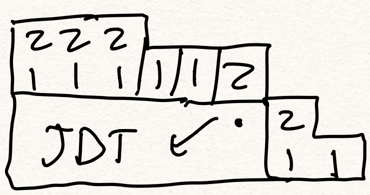


RSK($\underbrace{\underline{222111112}\ 211}_{\text{ballot? yes}}$)

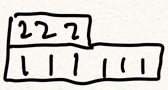
222
111112 ← 211

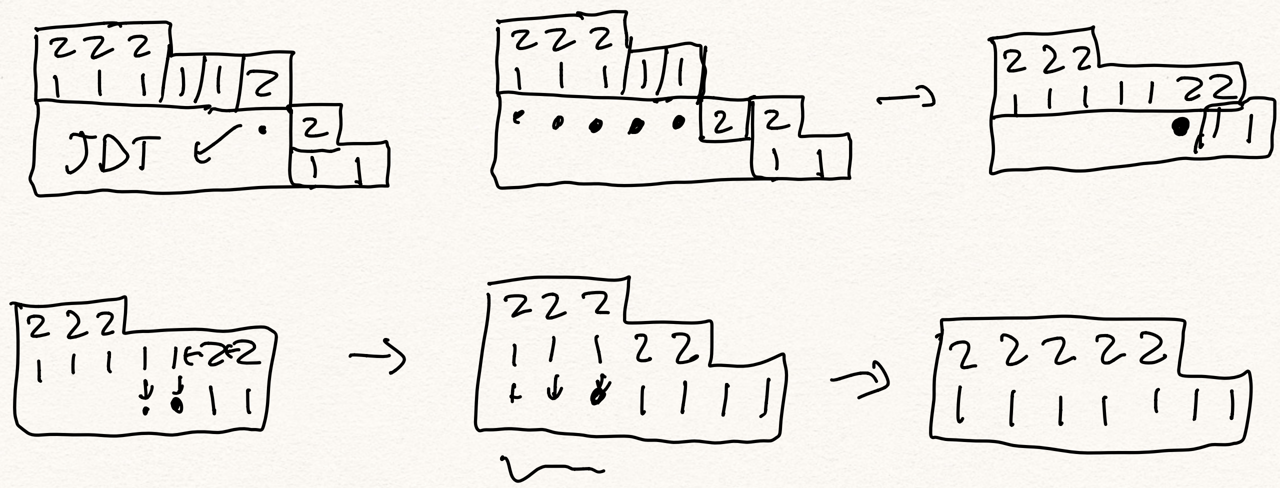
$$\boxed{V_\mu \otimes V_\lambda = \bigoplus_{|\nu| = |\mu| + |\lambda|} c_{\mu\lambda}^{\nu} V_\nu}$$

Alt rule:



$c_{\mu\lambda}^{\nu}$ = # pairs S, T of SSYT's of shapes μ, λ (1's, 2's) s.t. $S \leftarrow rw(T)$ is highest wt tab shape ν



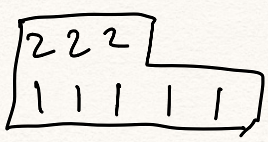


Connection to Schur functions

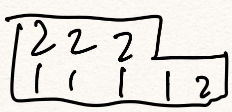
Consider $\quad s_\lambda(x_1,x_2) = \sum\limits_{\substack{\text{SSYT } T \\ \text{of shape } \lambda \\ \text{w/ 1's, 2's}}} \underline{x_1^{\#1\text{'s}}\, x_2^{\#2\text{'s}}}$.

Mult: $\quad s_\mu(x_1,x_2)\cdot s_\lambda(x_1,x_2) = \sum \underline{c^{\nu}_{\lambda\mu}}\, s_\nu$

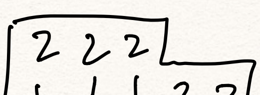
<u>Thm:</u> $\quad s_\lambda(q,q^{-1})$ is the weight gen. fn. of $\underline{V_\lambda}$. i.e. $s_\lambda(q,q^{-1}) = \sum\limits_{\substack{\text{weights} \\ \alpha}} q^{wt(v_{\underline{\alpha}})}$

 $\quad x_1^5\, x_2^3 \xrightarrow[x_2=q^{-1}]{x_1=q} q^2$

$\downarrow$

 $\quad x_1^4\, x_2^4 \longrightarrow q^0$

$\downarrow$

 $\quad \longrightarrow q^{-2}$

Adjoint rep of $\mathfrak{sl}_2$

Def: Adj. rep of Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} \to \underbrace{\mathfrak{gl}(\mathfrak{g})}$$
$$X \to (Y \mapsto [X,Y])$$

(action of $\mathfrak{g}$ on itself via the Lie bracket).

E, F, H acting on E, F, H:

$$[H, E] = 2E$$
$$[H, F] = -2F$$
$$[H, H] = 0$$

$$\begin{tikzcd}
2 \bullet \langle E \rangle \arrow[d] \\
0 \bullet \langle H \rangle \arrow[d] \\
-2 \bullet \langle F \rangle
\end{tikzcd}$$

Adj rep of $\mathfrak{sl}_2$.

$\mathfrak{sl}_3$ Next time