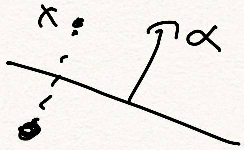
Recall: $r_\alpha(x) = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle} \cdot \alpha$



"reflection of x across hyperplane w/ normal vector α"

Def: A $\overset{\text{(abstract)}}{\wedge}$ root system in a real inner product space V is a nonempty finite set R of nonzero vectors spanning V such that

1. $r_\alpha(\beta) \in R$ for all $\alpha, \beta \in R$.
2. $\frac{2\langle \alpha, \beta\rangle}{\langle \beta, \beta\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in R$.
3. If $\beta \in R$ is a scalar multiple of $\alpha \in R$, then $\beta = \pm\alpha$.

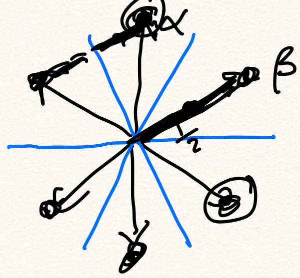
Thm: R is the set of roots of a semisimple Lie algebra with $\mathfrak{h}^* = V$ iff R is an (abstract) root system.

Examples of root systems

Ex: • $\mathfrak{sl}_n$: $R = \{ L_i - L_j \text{ for } \underset{i,j \in \{1,\dots,n\}}{i \neq j} \}$ (type A_n root system)

Simple roots: $L_i - L_{i+1}$ $(i = 1, \dots, n-1)$

sl_3:



Recall: if $\vec{v}, \vec{w} \in \mathbb{R}^n$, $|\vec{w}| = 1$

$$\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}} = \vec{v}\cdot\vec{w} = \text{length of projection of } \vec{v} \text{ onto } \vec{w}$$

$$\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} = 1$$

$$r_\beta(\alpha) = \alpha - \frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\beta$$

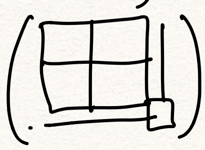
<u>Ex:</u> so_{2n+1} $\boxed{L_1, \dots, L_n \text{ basis}}$

SO_n = rotation group = special orthogonal gp $= \{A : AA^T = I, \det A = 1\}$

$\underline{R = \{\pm L_i \pm L_j, \pm L_i\}}$

Simple roots:

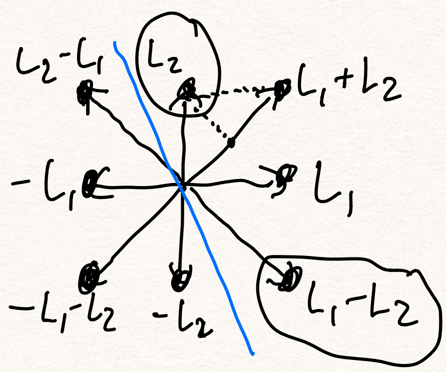
$\{L_1 - L_2, L_2 - L_3, \dots, L_{n-1} - L_n, \underline{L_n}\}$



$\boxed{so_n : \{X : X + X^T = 0, \ \underline{tr}\, X = 0\}}$

(Hwk: how to deduce root system of so_{2n+1})

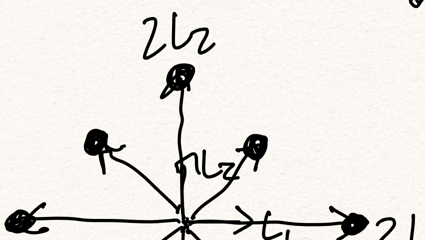
so_5: $(n=2)$ $\pm L_1 \pm L_2, \pm L_1, \pm L_2$

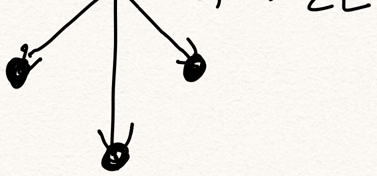


(type B_n)

<u>Ex:</u> sp_{2n}: $R = \{\pm L_i \pm L_j, \pm 2L_i\}$ (type C_n)

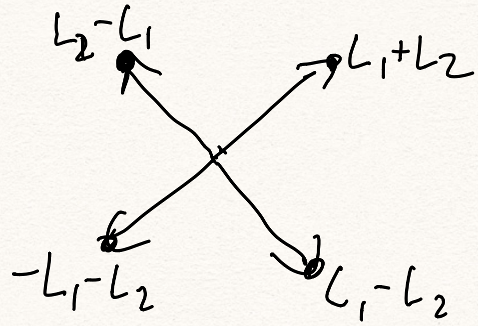
sp_4:



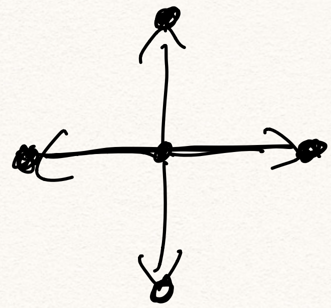


Ex: so_{2n}: $R = \{\pm L_i \pm L_j\}$ (Type D_n)

so_4:



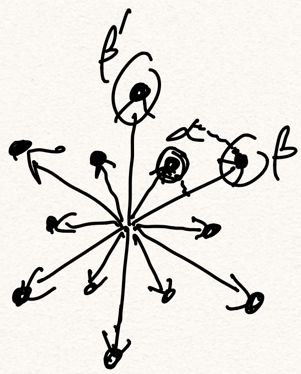
$\cong$



$sl_2 \times sl_2$

$so_4 \cong sl_2 \times sl_2$ as Lie algebras.

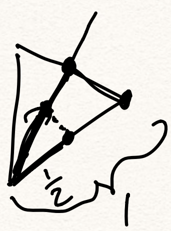
Ex: Type G2:



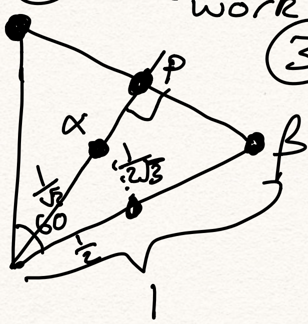
① Closed under reflections? ✓

② Projection lengths work ✓

③ ✓



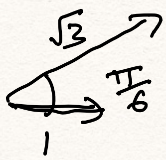
α: center of $\Delta 0, \beta, \beta'$



$p = \frac{3}{2}\alpha$

Lemma: In a root system R, the angle between any two roots is one of:

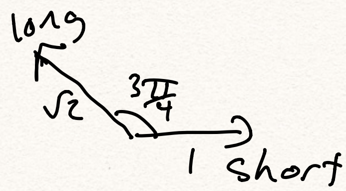
- $\frac{\pi}{6}$:

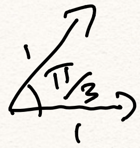


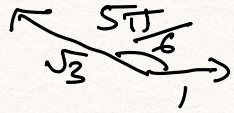
- $\frac{2\pi}{3}$

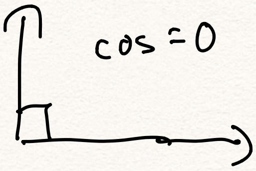
- $\frac{\pi}{4}$:

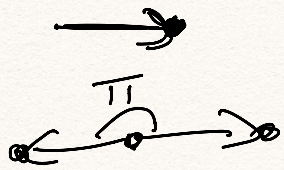
- $\frac{3\pi}{4}$



- $\frac{\pi}{3}$: 

- $\frac{5\pi}{6}$ 

- $\frac{\pi}{2}$:  $\cos = 0$

- 0:
- π: 

$\Big\}\ \cos = \pm 1$

Pf: $\dfrac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle} = \dfrac{2\cos\theta\,|\alpha||\beta|}{|\beta|^2} = 2\cos\theta\,\dfrac{|\alpha|}{|\beta|} \in \mathbb{Z}$

Similarly, $2\cos\theta\,\dfrac{|\beta|}{|\alpha|} \in \mathbb{Z}$

Multiply: $4\cos^2\theta \in \mathbb{Z}$ $\quad 0 \le \cos^2\theta \le 1$

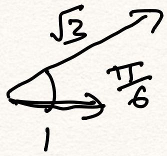
so $\quad 4\cos^2\theta \in \{0,1,2,3,4\}$

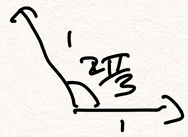
$\Rightarrow \cos\theta = \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1, 0.$

This gives possibilities for θ,
lengths follow from $\dfrac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$.

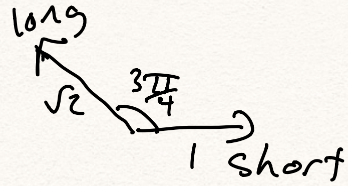
Recall:

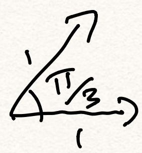
Lemma: In a root system R, the angle between any two roots is one of:

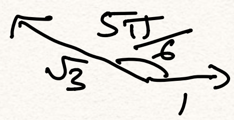
- $\frac{\pi}{6}$: 

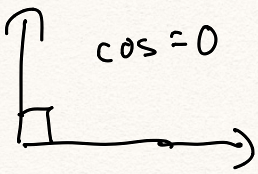
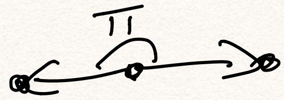
- $\frac{2\pi}{3}$ 

- $\frac{\pi}{4}$: 

- $\frac{3\pi}{4}$ 

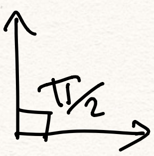
- $\frac{\pi}{3}$: 

- $\frac{5\pi}{6}$ 

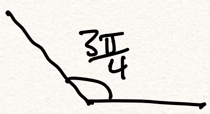
- $\frac{\pi}{2}$:  $\cos = 0$
- 0: 
- π: 

$\Big\}\ \cos = \pm 1$

Cor: The angle between any two distinct <u>simple roots</u> is one of:

- 

- 

,
- 

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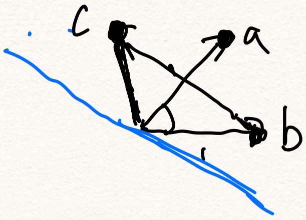
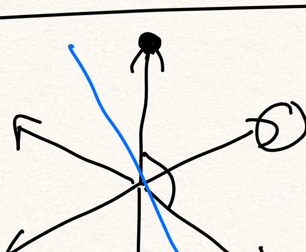
Pf: Assume for $\rightarrow\leftarrow$ we have two simple roots at an acute angle:



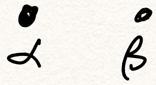
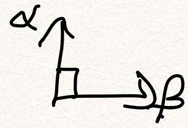
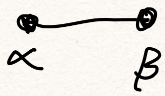
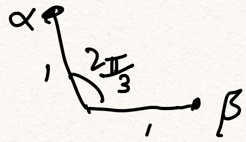
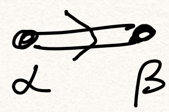
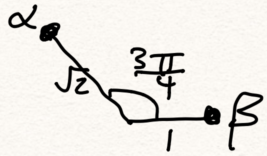
$a = c + b$

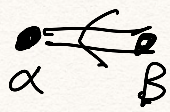
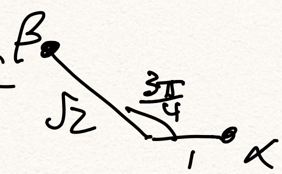
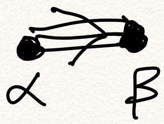
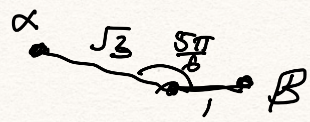
a not simple

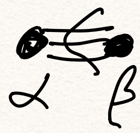
$\rightarrow\leftarrow$

Ex: sl_3: 

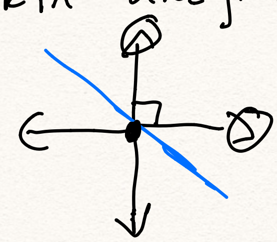
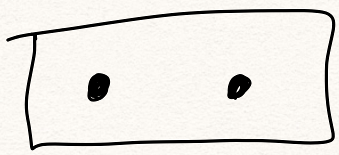
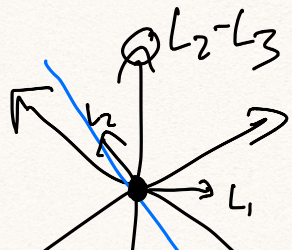
Def: Dynkin diagram of R: Graph w/

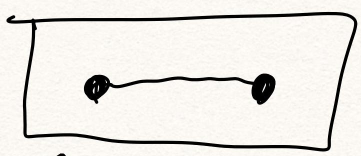
- **Nodes:** simple roots of R
- **Edges:** edge between α, β if angle θ btwn α, β is $> 90°$
  - No edge:  if 
  - One edge:  if 
  - Two edges:  if 

     if 
  - Three edges:  if 

    similarly 

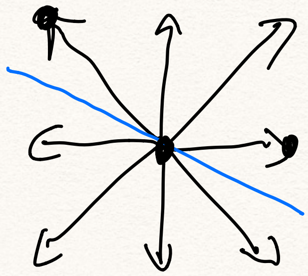
Ex: Dynkin diagram of:

-  $(sl_2 \times sl_2 \cong so_4)$ 
- 

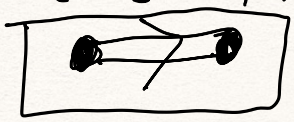


(sl_3)

•



$(so_5 \cong sp_4)$



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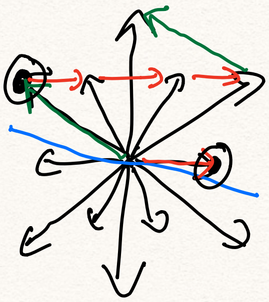
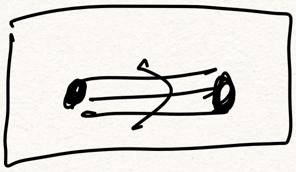




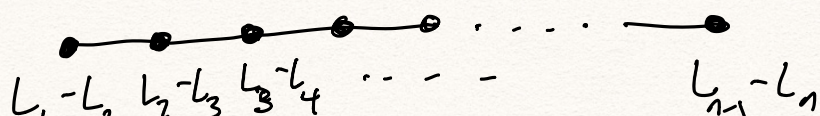
g_2

$\dim = 12 + 2 = 14$

<u>Goal</u>: Classify root systems by classifying Dynkin diagrams.

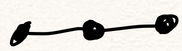
<u>Thm</u>: The possible connected Dynkin diagrams for a root system of a s.s. Lie alg are:

① A_n



(sl_n root systems)

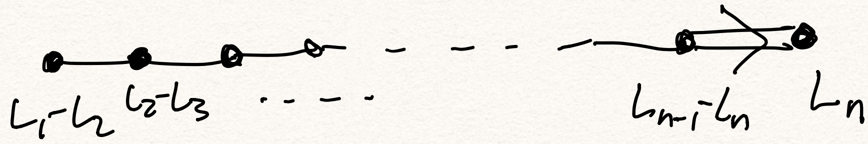
angle $\frac{2\pi}{3}$.



: sl_4

$sl_4 \cong so_6$

② B_n:



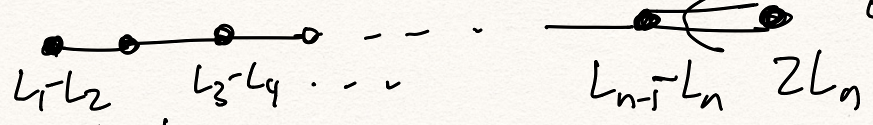
(so_{2n+1})

$\frac{3\pi}{4}$ angle

$so_5 \cong sp_4$

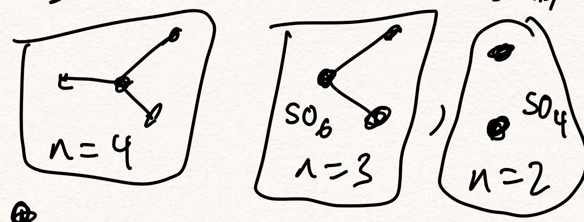
③ C_n:

(sp_{2n})

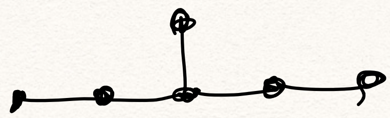


④ D_n: $(n \geq 3)$

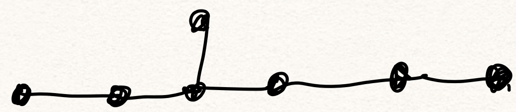
(so_{2n})



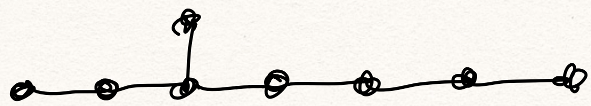
⑤ E_6:



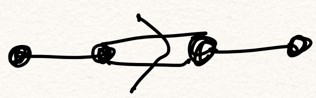
⑥ E_7:



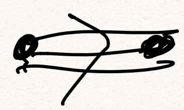
⑦ E_8:



⑧ F_4:



⑨ G_2:



"exceptional types"

<u>Def</u>: An <u>admissible diagram</u> (Coxeter diagram) is a graph of <u>n nodes</u> representing <u>n independent unit</u> vectors $e_1, \dots, e_n$, w/ angle between e_i, e_j being:

$$\pi/2, \quad \frac{2\pi}{3}, \quad \frac{3\pi}{4}, \text{ or } \frac{5\pi}{6},$$

$e_i \quad e_j \qquad e_i \quad e_j \qquad e_i \quad e_j \qquad e_i \quad e_j$

(no directionality on edges)

We first classify connected admissible diagrams

<u>Note</u>: The underlying <u>undirected</u> graph of any Dynkin diagram is an admissible graph.

<u>Lemma 1</u>: Any vertex-induced subgraph of an admissible diagram is admissible.

<u>Pf</u>: Deleting some vectors still results in a set of linearly indep. vectors w/ angles $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$, or $\frac{5\pi}{6}$. $\square$

<u>Lemma 2</u>: At most $n-1$ pairs of nodes are connected by lines.

<u>Pf</u>: If e_i, e_j are connected then

$$2\langle e_i, e_j\rangle \leq -1$$

$$\left\langle \sum e_i, \sum e_i \right\rangle > 0$$

$$\underbrace{\sum_{i=1}^{n} \underbrace{\langle e_i, e_i\rangle}}_{} + 2\sum_{1\leq i<j\leq n} \langle e_i, e_j\rangle > 0$$

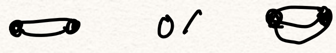
$$n + \sum 2\langle e_i, e_j\rangle > 0$$

$=$ each of these is ≤ -1 if not 0

So # of nonzero inner products $\langle e_i, e_j \rangle$ is at most $n-1$.

Therefore at most $n-1$ pairs are connected. $\square$

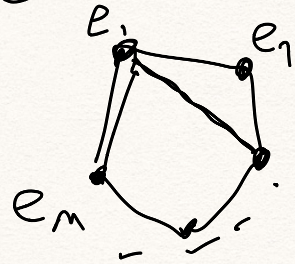
<u>Lemma 3</u>: An admissible diagram has no cycles (other than those in



or

)

<u>Pf</u>: If it had a cycle, $e_{i_1}, \dots, e_{i_m}$



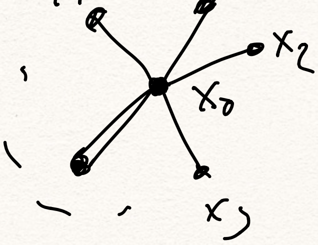
then restrict to that cycle; by Lemma 1 this is admissible.

At least m pairs are connected, contradicting Lemma 2. QED

<u>Lemma 4</u>: No node has degree ≥ 4.

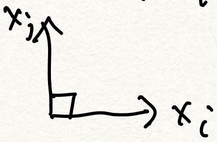
<u>Proof</u>: Consider subgraph induced by one node x_0 and its neighbors $x_1, \dots, x_r$.

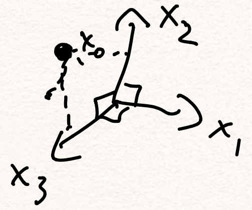
Since the graph is a tree, x_i, x_j not connected for

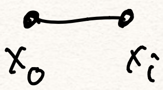


$i \neq j \in \{1, 2, \dots, r\}$

$\Rightarrow \langle x_i, x_j \rangle = 0$ for $i \neq j \in \{1, \dots, r\}$.

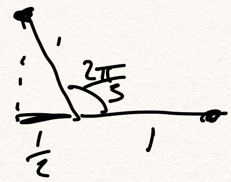


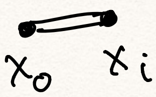


If 

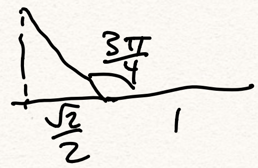
: $\langle x_0, x_i \rangle = -\frac{1}{2}$

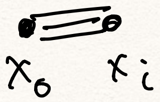
$\langle x_0, x_i \rangle^2 = \frac{1}{4}$



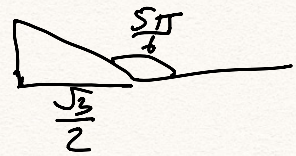
If 

: $\langle x_0, x_i \rangle^2 = \frac{1}{2} = \frac{2}{4}$



If 

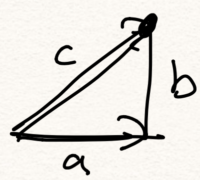
: $\langle x_0, x_i \rangle^2 = \frac{3}{4}$



$$\Rightarrow \sum_{i=1}^{r} 4 \langle x_0, x_i \rangle^2 = \text{deg of } x_0.$$

Since $x_1, \dots, x_r$ orthogonal, x_0 not in span of $x_1, \dots, x_r$ (by assumption)

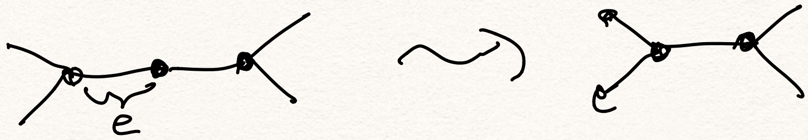
$$1 = \langle x_0, x_0 \rangle^2 > \sum_{i=1}^{r} \langle x_0, x_i \rangle^2$$



$$\deg(x_0) = 4 \sum_{i=1}^{r} \langle x_0, x_i \rangle^2 < 4$$

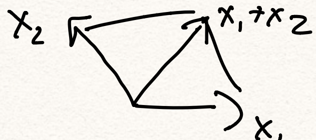
QED.

Lemma 5: Contracting a single edge (not part of a double or triple edge) in an adm diagram results in an admissible diagram.



Pf: 

Since x_1, x_2 have a single edge, their angle is $\frac{2\pi}{3}$.



So x_1+x_2 is a unit vector; replace x_1, x_2 w/ x_1+x_2.

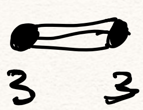
Suppose x_j is another vertex. x_j connected to at most one of x_1, x_2 (otherwise cycle)

Say x_1.

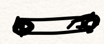
$$\langle x_j, x_1+x_2\rangle = \langle x_j, x_1\rangle + \cancel{\langle x_j, x_2\rangle}^{\,0}$$
$$= \langle x_j, x_1\rangle$$

QED

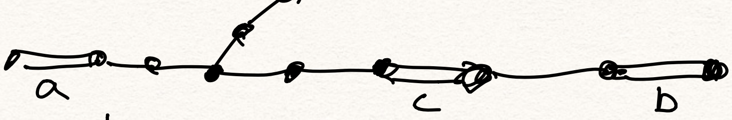
Cor 1: If a triple edge exists, the graph is G2.



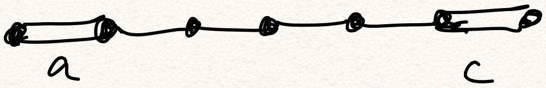
(G2)

Cor 2: Any connected admissible diagram has at most one double edge .

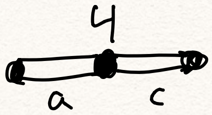
Pf: Assume ≥ 2 double edges



Find a path from one to another



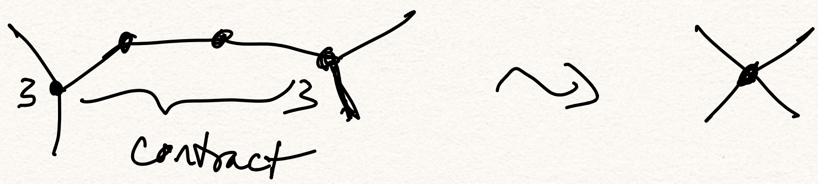
Restricting to this path gives an admissible diagram by Lemma 1. Contracting each of the single edges btwn a, c gives:



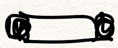
Admissible by Lemma 5
Not admissible by Lemma 4
(deg ≤ 3)

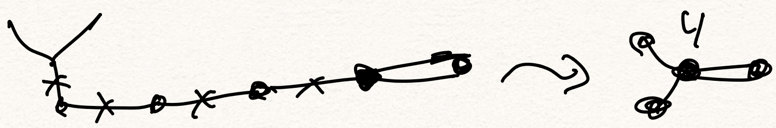
$\rightarrow\leftarrow$

Cor 2(b): Any connected admissible diagram has at most one triple vertex

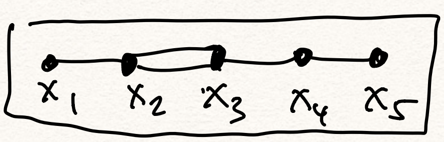


Cor 2(c): Can't have both  and



(Otherwise: 

).

Lemma 6: 

is not admissible.

Pf: We'll find v, w that violate:

$$\langle v, w\rangle^2 < |v|^2 \cdot |w|^2 \qquad (v, w \text{ not scalar multiples})$$

Define $v = x_1 + 2x_2$, $w = 3x_3 + 2x_4 + x_5$ $\Big\}$ v, w independent

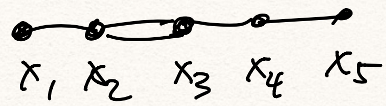
Then $\langle v, w\rangle^2 = \left(\langle x_1 + 2x_2, 3x_3 + 2x_4 + x_5\rangle\right)^2$

$$= \left(\langle 2x_2, 3x_3\rangle\right)^2$$

$$= \left(6\langle x_2, x_3\rangle\right)^2$$

$$= \left(6 \cdot \frac{-\sqrt{2}}{2}\right)^2$$

$$= 18$$



$|v|^2 = \langle v, v\rangle = \langle x_1 + 2x_2, x_1 + 2x_2\rangle$

$$= \langle x_1, x_1\rangle + 4\langle x_1, x_2\rangle + 4\langle x_2, x_2\rangle$$

$$= 1 + 4\left(\frac{-1}{2}\right) + 4 \cdot 1$$

$$= 3$$

$|w|^2 = \langle w, w\rangle = \langle 3x_3 + 2x_4 + x_5, 3x_3 + 2x_4 + x_5\rangle$

$$= 9 + 4 + 1 + 12\langle x_3, x_4\rangle + 4\langle x_4, x_5\rangle$$

$$= 14 + 12\left(-\frac{1}{2}\right) + 4\left(-\frac{1}{2}\right)$$

$$= 14 - 6 - 2$$

$$= 6$$

$|v|^2|w|^2 = 3 \cdot 6 = 18 = \langle v, w\rangle^2$ $\rightarrow\leftarrow$ to Cauchy-Schwarz.

QED.

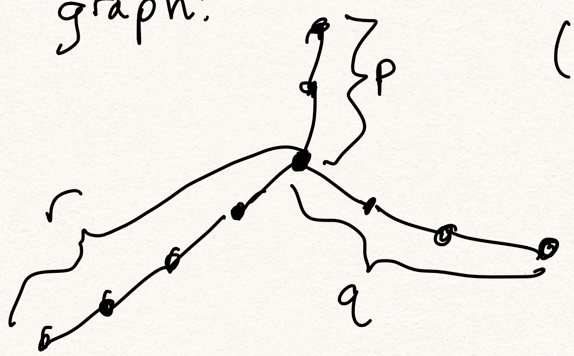
Cor: Only possible graphs w/ are:

①  (F_2)

②  (B_n, C_n)

(Finished , , what happens w/ only ?)

Lemma 7: Let p, q, r be lengths of legs from a trivalent node in an admissible graph:



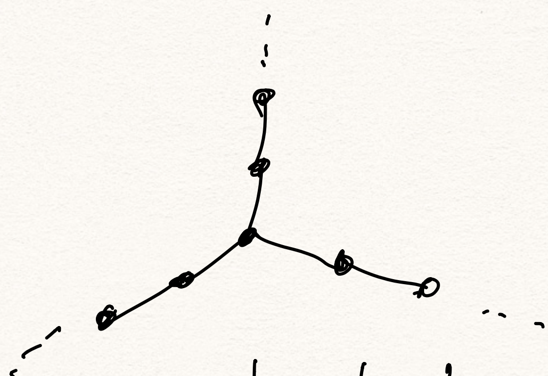
$(p=3, q=4, r=5)$.

Then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

(Pf next time)

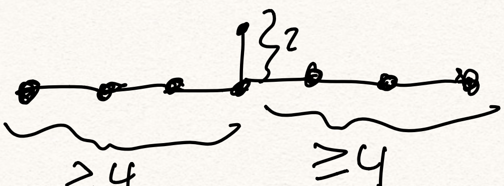
Conclusion:

Cor 4: Can't have 2 edges in each direction

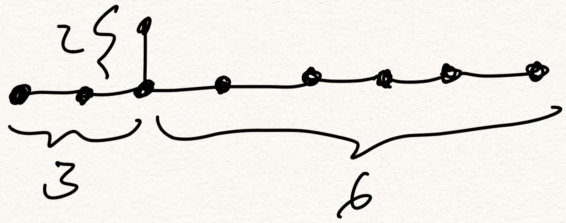


$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

Cor 5: Can't have:

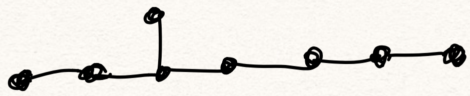


$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

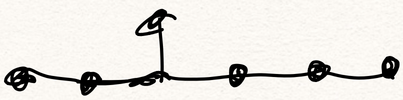
OR 

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$

<u>Cor 6</u>: Only trivalent possibilities:

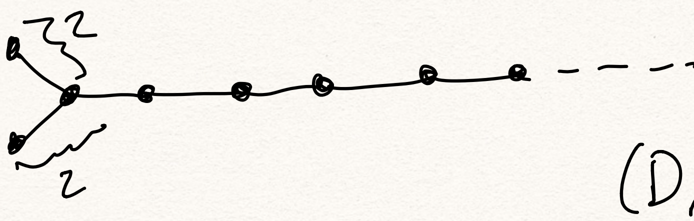


(E8)



(E7)

(E6)



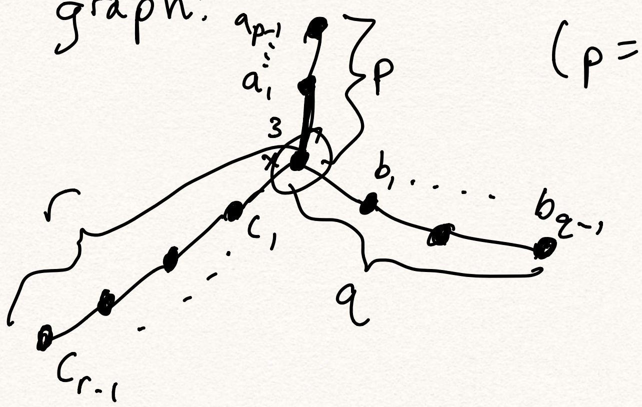
$\frac{1}{2}+\frac{1}{2}+\frac{1}{N}>1$

(D_n)

Only other graph:

(A_n)

Lemma 7: Let p, q, r be lengths of legs from a trivalent node in an admissible graph:



$(p=3, q=4, r=5)$.

Then $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$.

Proof: Consider:

$$\underline{u} := \frac{(p-1)a_1+(p-2)a_2+\cdots+a_{p-1}}{\sqrt{\frac{p(p-1)}{2}}} \Big\} \text{ unit vector}$$

$$v := \frac{(q-1)b_1+(q-2)b_2+\cdots+b_{q-1}}{\sqrt{\frac{q(q-1)}{2}}} \Big\} \quad "$$

$$w := \frac{(r-1)c_1+(r-2)c_2+\cdots+c_{r-1}}{\sqrt{\frac{r(r-1)}{2}}} \Big\} \quad "$$

u, v, w orthogonal unit vectors,

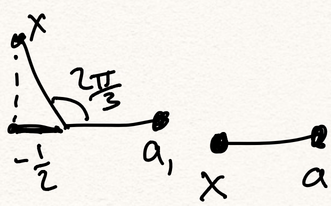
$\underline{x}$ not in their span, not orth to u, v, w.

by assumption

$$\boxed{1 = |x|^2 > \langle x,u\rangle^2 + \langle x,v\rangle^2 + \langle x,w\rangle^2} \quad (*)$$

$$\langle x,u\rangle^2 = \left(\frac{\langle x,a_1\rangle (p-1)}{\sqrt{\frac{p(p-1)}{2}}}\right)^2 = \frac{\langle x,a_1\rangle^2 (p-1)\cdot 2}{p} = \frac{1}{4}\,\frac{2(p-1)}{p}$$

$$= \boxed{\frac{1}{2}\left(1-\frac{1}{p}\right)}$$



$$\langle x, v\rangle^2 = \frac{1}{2}\left(1-\frac{1}{q}\right)$$

$$\langle x, w\rangle^2 = \frac{1}{2}\left(1-\frac{1}{r}\right)$$

Substitute into $(*)$:

$$1 > \frac{1}{2}\left(1-\frac{1}{p}\right) + \frac{1}{2}\left(1-\frac{1}{q}\right) + \frac{1}{2}\left(1-\frac{1}{r}\right)$$

$$1 > \frac{1}{2}\left(3-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\right)$$

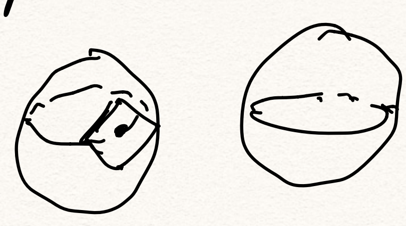
$$2 > 3 - \left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$$

$$\boxed{\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1}$$

QED.

<u>Summary</u>: We've classified the Dynkin diagrams (type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2)

Therefore classified root systems, semisimple Lie algebras, connected compact Lie groups



<u>Application</u>: classification of finite <u>simple</u> <u>groups</u>: Groups w/ no proper nontrivial normal subgroup.

Big categories:
- Symmetric, S_n
- Alternating A_n
- Cyclic groups C_n
- $\boxed{\text{Groups of Lie type (Chevalley gps)}}$
- Exceptional (27)

Group of Lie type: constructed from Lie groups over $\mathbb{F}_q$.

Ex: $SL_n(\mathbb{F}_q) = \{ n \times n$ matrices M over $\mathbb{F}_q$ w/ $\det M = 1 \}$

Crystals in other Lie types:

Recall: In type A, the standard crystal was crystal of $\mathfrak{sl}_n$ acting on $\mathbb{C}^n$ $(V^{\square})$

$$\mathfrak{sl}_n \to \mathfrak{gl}_n$$
$$X \mapsto X$$

$$\boxed{1} \xrightarrow{f_1} \boxed{2} \xrightarrow{f_2} \boxed{3} \xrightarrow{f_3} \boxed{4} \cdots \xrightarrow{f_n} \boxed{n}$$

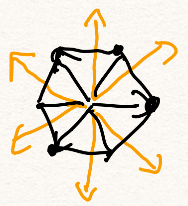
We took tensor products of $V^{\square}$ with itself to generate (not nec. connected) word crystals. (connected components $\leftrightarrow$ irred. reps).

Def: (Combinatorial crystals) (Kashiwara).

Let R be a root system w/ simple roots $\alpha_1, \ldots, \alpha_n$, weight lattice Λ

$$\{\beta : \frac{2\langle \beta, \alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

sl_3:



A (finite seminormal) <u>crystal</u>: is a finite set $\underline{\mathcal{B}}$ along with maps:

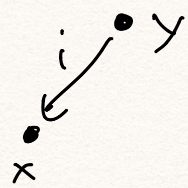
- $\underline{e_i}, \underline{f_i} : \mathcal{B} \to \mathcal{B} \cup \{0\}$ for $i = 1, \dots, n$
- $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ for $i = 1, \dots, n$
- $wt : \mathcal{B} \to \Lambda$

s.t.,

(K0) (seminormal) $\varepsilon_i(x) = \max\{k : e_i^k(x) \neq 0\}$
$\varphi_i(x) = \max\{k : f_i^k(x) \neq 0\}$

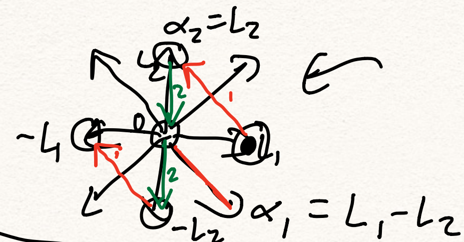
(K1) If $x, y \in \mathcal{B}$, $e_i(x) = y$ iff $f_i(y) = x$

In this case,
$wt(y) = wt(x) + \alpha_i$



(K2): $\varphi_i(x) - \varepsilon_i(x) = \dfrac{2\langle wt(x), \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}$. (type A: $wt(x)_i - wt(x)_{i+1}$)

<u>Ex</u>: type B_2 $\alpha_1 \Rightarrow \alpha_2$



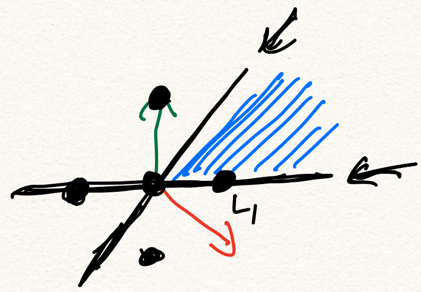
so_5 : acts on 5-dim'l $\mathbb{C}^5$.

<u>Crystal graph</u>: (Standard for B_2)

$$\underset{wt=L_1}{\bullet} \xrightarrow{} \underset{L_2}{\bullet} \xrightarrow{} \underset{0}{\bullet} \xrightarrow{} \underset{-L_2}{\bullet} \xrightarrow{} \underset{-L_1}{\bullet}$$

$$\underset{L_1}{\boxed{1}} \longrightarrow \underset{L_2}{\boxed{2}} \longrightarrow \underset{0}{\boxed{0}} \longrightarrow \underset{-L_2}{\boxed{\bar{2}}} \longrightarrow \underset{-L_1}{\boxed{\bar{1}}}$$


5
5


L_1