**Recall:** Weyl group = group of reflections stabilizing the root system for $\mathfrak{g}$

$\Rightarrow$ $W \curvearrowright \mathfrak{h}^*$

Simple reflection $s_i = s_{\alpha_i}$ ($\alpha_i$ simple root)

$$s_i(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

$W$ gen by $s_i$'s.

**Lemma:** $W$ acts on the set of weights of any representation $V$ of $\mathfrak{g}$.

**Pf:** We show each $s_i$ acts on the set of weights.

Recall $V$ decomposes into weight spaces $V = \oplus V_\beta$, symmetric about perpendicular to any $\alpha_i$ because if we restrict $V$ to a rep of $(\mathfrak{sl}_2)_{\alpha_i} = \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}]$, we obtain symmetric $\mathfrak{sl}_2$ strings.

Thus the action can be defined by $s_i$ mapping $\beta$ to $s_i(\beta)$, which is also a weight of $V$. $\square$

**Fact:** In associated Lie group $G$,

$W = N_G(T)/T$ where

- $T$ is torus
- $N_G(T) = \{g : gTg^{-1} = T\}$

Ex: $G = SL_n$, $T = \{\text{diagonal matrices w/ } \det = 1\}$

$N_G(T)$: When is $gtg^{-1}$ diagonal?

Want: $gtg^{-1} e_i = \lambda e_i$ for some $\lambda$

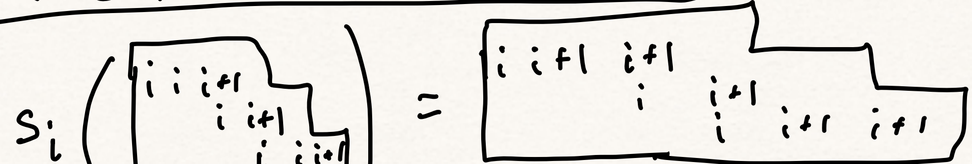
$tg^{-1}e_i = \lambda g^{-1} e_i$

so $\underset{\substack{\| \\ \mu e_j}}{g^{-1}e_i} \in \{e_1, \dots, e_n\}$ or scalar mult thereof

i.e. $g$ is a permutation matrix w/ each entry scaled: $\begin{pmatrix} 0 & 3 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$, $\det = 1$

So $N_G(T)/T$ = subgp of permutation matrices $= S_n$.

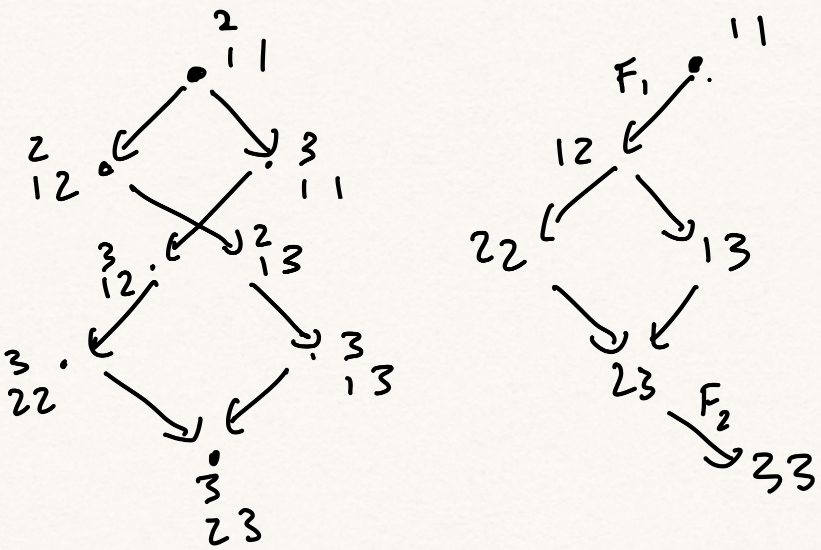
Action on tableau crystals:




i i i+1 i i+1 i i i+1

- bracket i's and i+1's
- Consider unbracketed word, $i^a (i+1)^b$
- Replace with $(i)^b (i+1)^a$.
- Output is $s_i(T)$.

# Vertical symmetry and longest word




<u>Recall:</u> Every connected crystal has a unique lowest weight and highest weight elt.

<u>Lemma:</u> Let $w_0$ = longest word in $S_n$.

$$\overset{\parallel}{s_1 s_2 s_1} = s_2 s_1 s_2 = n, n-1, n-2, \ldots, 1.$$

In general: $s_1 (s_2 s_1)(s_3 s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_2 s_1)$

Then $w_0$ (highest wt) = (lowest wt).

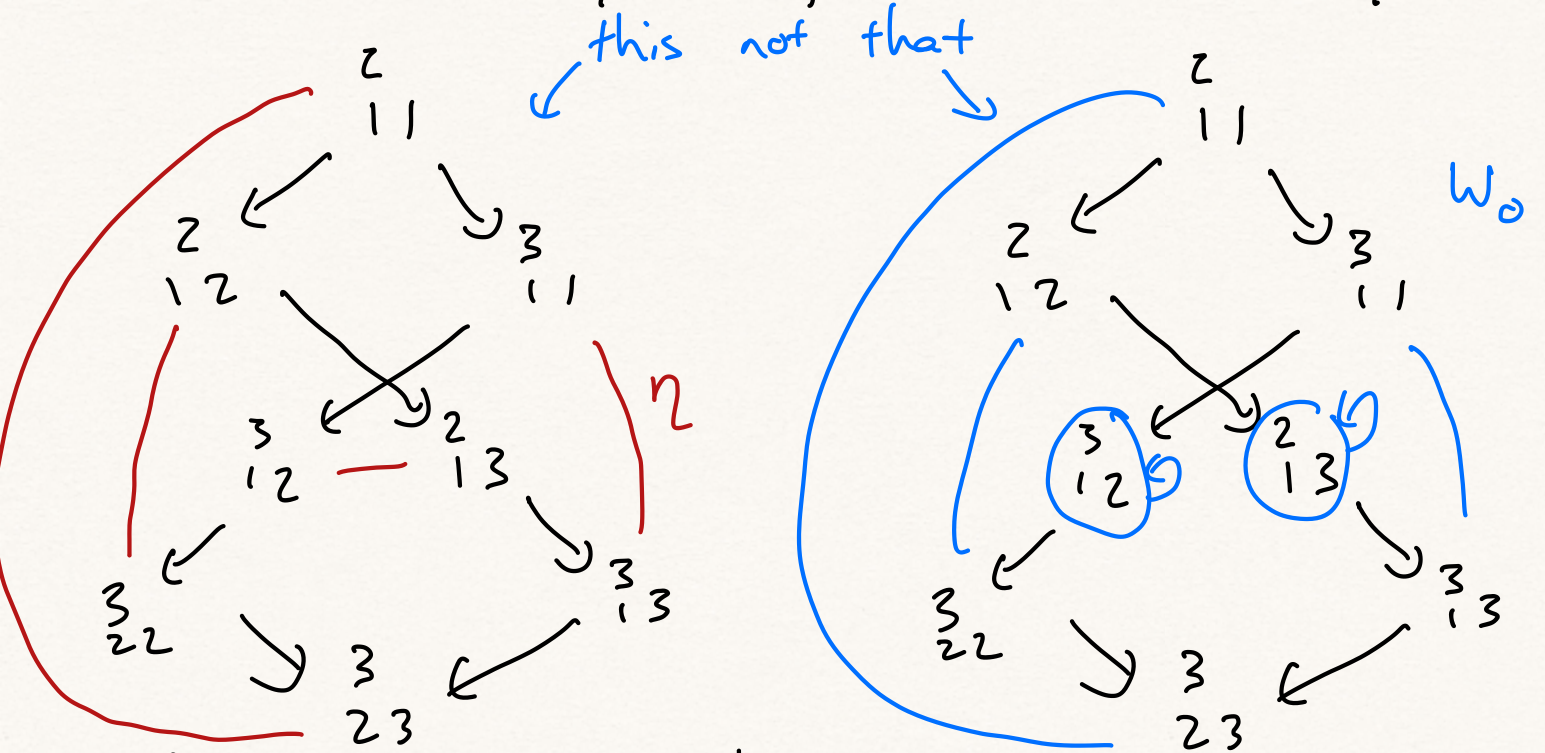
<u>Pf:</u> h.w tableau is:

$$T_h = \begin{matrix} 44 \\ 33333 \\ 22222 \\ 1111111 \end{matrix} \xrightarrow{s_3 s_2 s_1} \begin{matrix} 44 \\ 33444 \\ 22222 \\ 1111144 \end{matrix}$$

$$\xrightarrow{s_2 s_1} \begin{matrix} 44 \\ 33444 \\ 22333 \\ 1111144 \end{matrix}$$

$$\xrightarrow{s_1} \begin{matrix} 44 \\ 33444 \\ 22333 \\ 1122244 \end{matrix} = T_\ell \qquad \square$$

Want vertical symmetry of entire crystal:




## <u>Schützenberger involution</u> $\eta$

### <u>Evacuation</u>:

- Turn 180°
- Reverse $i$'s w/ $n-i$'s
- JDT

<u>OR</u>: • Take out an $i$, do a JDT slide, fill in w/ $n-i$, etc.

$$\begin{array}{l} 3 \\ 2\,3\,4 \\ 1\,1\,2\,3 \end{array} \xrightarrow{180^\circ} \begin{array}{r} 3\,2\,1\,1 \\ 4\,3\,2 \\ 3 \end{array} \xrightarrow{\text{flip}} \begin{array}{r} 2\,3\,4\,4 \\ 1\,2\,3 \\ 2 \end{array} \xrightarrow{\text{JDT}} \begin{array}{l} 4 \\ 2\,3\,3 \\ 1\,2\,2\,4 \end{array}$$

$$\underbrace{\begin{array}{l}3\\234\\1123\end{array} \xrightarrow{s_3 s_2 s_1} \begin{array}{l}3\\244\\1122\end{array} \xrightarrow{s_2 s_1} \begin{array}{l}3\\244\\1113\end{array} \xrightarrow{s_1} \begin{array}{l}3\\244\\1223\end{array}}_{w_0}$$

NOT the same!

(Also not the same on (2,1) crystal).

<u>Thm (vertical symmetry)</u>: Let $T$ be any tableau for $sl_n$ and suppose $F_{i_1} \cdots F_{i_r} T_h = T$.

highest weight

Then $E_{n-i_1} \cdots E_{n-i_r} T_L = \eta T$. In other words, the longest word $w_0$ gives vertical symmetry of the crystal, with respect to exchanging the $i$ arrows with $n-i$ arrows.

(Berenstein - Zelevinsky) (difficult!)

Specifically:

- $\eta^2 = 1$ (involution)
- $\mathrm{wt}(\eta X) = \mathrm{rev}(\mathrm{wt}(X))$
- $\eta T_h = T_L$
- $F_i(X) = X' \iff E_{n-i}(\eta X) = \eta X'$
- $\varphi_i(\eta X) = \varepsilon_i(X)$ and $\varepsilon_i(\eta X) = \varphi_i(X)$