General Lie algebra rep. theory

Let $\mathfrak{g}$ be a complex Lie algebra

$\mathfrak{g}$ has an adjoint representation

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$
$$x \longmapsto [x, -]$$

$\mathfrak{gl}_n \cong$ if $V \cong \mathbb{C}^n$

$\mathfrak{gl}(V) =$ Lie alg. of $GL(V)$

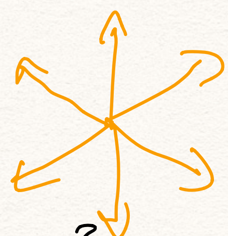
$\boxed{\mathrm{ad}(\mathfrak{sl}_2)}$

Def: The roots of $\mathfrak{g}$ are the weights α st. the weight space $\mathfrak{g}_\alpha$ of ad. is nonzero.

Def: $\underline{\underline{\mathfrak{h}}} \subseteq \mathfrak{g}$ is the maximal diagonalizable subalgebra (Cartan)

Def: $\mathfrak{g}$ is semisimple if the roots span $\underline{\underline{\mathfrak{h}^*}}$.

$\dim \mathfrak{h}^* = \dim \mathfrak{h}$



Non-ex: $\mathfrak{gl}_n = \{\text{all } n \times n \text{ matrices}\}$

roots: same as $\mathfrak{sl}_n$: $\underbrace{L_i - L_j}$ $(i \neq j)$

$$\begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_n \end{pmatrix}$$

But $\mathfrak{h} \subseteq \mathfrak{gl}_n$
$=$ diagonal matrices (not $\mathrm{tr} = 0$)
is one higher dim than in $\mathfrak{sl}_n$!

$\mathfrak{gl}_n$ not semisimple

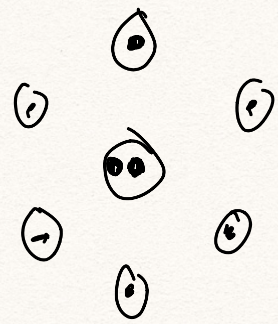
$\mathfrak{sl}_n$ is.

$\boxed{\text{From now on we assume } \mathfrak{g} \text{ is semisimple}}$

Facts: (1) $\mathfrak{g} = \underbrace{\mathfrak{h}}_{\substack{\text{Cartan}}} \oplus \bigoplus_{\alpha \in R} \underbrace{\mathfrak{g}_\alpha}_{\text{root space}}$

$\left(\mathfrak{sl}_3 = \underbrace{\langle H_1, H_2 \rangle} \oplus \bigoplus_{i \neq j} \langle E_{ij} \rangle \right)$

(2) Each $\mathfrak{g}_\alpha$ is 1-dim.

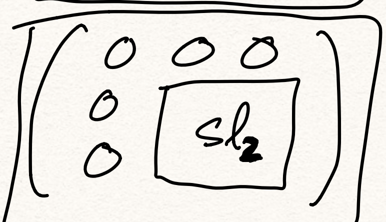


(3) If α is a root, so is $-\alpha$.

(4) Little $\mathfrak{sl}_2$'s: for each root.

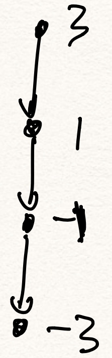
$$\mathfrak{sl}_2 \cong S_\alpha = \underbrace{\mathfrak{g}_\alpha} \oplus \underbrace{\mathfrak{g}_{-\alpha}} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}$$

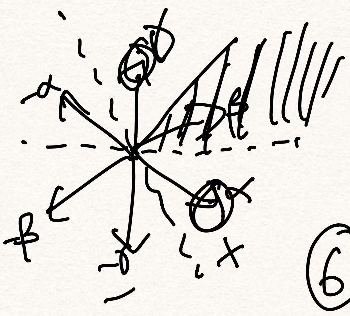
$$\begin{array}{ccc} \| & \| & \cap \\ & & \mathfrak{h} \\ & & \| \\ \langle E_\alpha \rangle & \langle F_\alpha \rangle & \langle H_\alpha \rangle. \\ E_{ij} & E_{ji} & \end{array}$$



(5) By $\mathfrak{sl}_2$-rep thy, every H_α has all integer eigenvalues.

Can decompose $\mathfrak{g}$ (as an s_α-rep) into <u>$\mathfrak{sl}_2$-strings</u> in the "α direction")





(6) Notice $\mathfrak{h}_{\mathbb{R}}^*$ is a vector space, all α's are in $\mathfrak{h}_{\mathbb{R}}^*$, choose a hyperplane in $\mathfrak{h}_{\mathbb{R}}^*$ to separate + and - roots.

<u>Def.</u> <u>simple root</u>: a positive root that is not the sum of other pos. roots.

The simple roots generate $\mathfrak{g}$ under $[,]$, +.

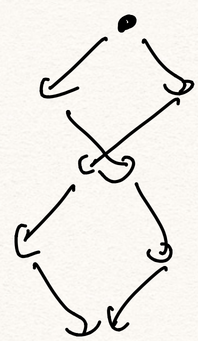
facts about reps of $\mathfrak{g}$

(7) Choose positive directions for each <u>simple root</u>

any <u>irred. rep.</u> of $\mathfrak{g}$ has a unique highest weight vector.

(8) Irred rep. V^β w/ highest weight β is generated by <u>F_α</u>'s starting from highest wt vector $v_\beta \in V^\beta$.

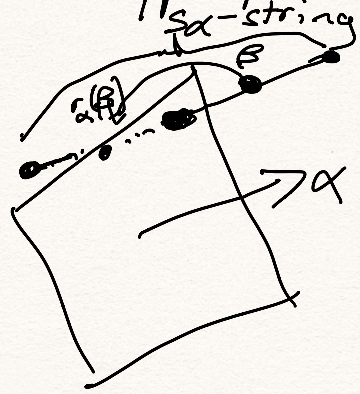
(In particular <u>$F_\alpha \cdot v_\beta$</u> is a weight vector of weight <u>$\beta - \alpha$</u>.)



Pf: $H \in \mathfrak{h}$, $H v_\beta = \beta(H) v_\beta$

$$
\begin{aligned}
H(F_\alpha v_\beta) &= (\underbrace{H F_\alpha - F_\alpha H} + F_\alpha H)\, v_\beta \\
&= ([H, F_\alpha] + F_\alpha H)\, v_\beta \\
&= -\alpha(H) F_\alpha v_\beta + F_\alpha \underline{H v_\beta} \\
&= -\alpha(H) F_\alpha v_\beta + F_\alpha \beta(H) v_\beta \\
&= -\alpha(H) F_\alpha v_\beta + \beta(H) F_\alpha v_\beta \\
&= \underbrace{(\beta(H) - \alpha(H))}_{(\beta-\alpha)(H)} (\underline{F_\alpha v_\beta})
\end{aligned}
$$

(9) Any rep. is symmetric under action of Weyl group W (group generated by reflections about hyperplanes normal to roots.)

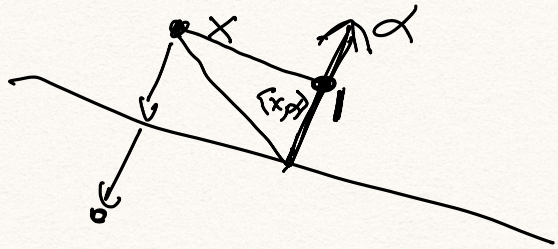


(10) Inner product on $\mathfrak{h}^*$: "Killing form":

$$\langle x, y \rangle = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y) : \mathfrak{g} \to \mathfrak{g})$$

Def: Reflection about hyperp. corresp to α is

$$\underline{r_\alpha(x)} = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \cdot \alpha.$$



Next time: define abstract root system as a set of vectors in an inner product space w/ some reflection properties.