Weyl Groups

Recall: Given a root system R, define s_α to be the reflection in the hyperplane $\perp$ to $\alpha \in R$

Def: $W =$ group gen. by s_α's.

Fact: Weyl group W generated by s_{α_i} for α_i a simple root.

Shorthand: $s_i := s_{\alpha_i}$

Fact: Weyl groups are Coxeter Groups:

- generated by $s_1, \dots, s_n$
- relations: (1) $s_i^2 = 1$ for all $i = 1, \dots, n$,
  (2) $(s_i s_j)^{m_{ij}} = 1$ for some m_{ij} for each $i, j \in \{1, 2, \dots n\}$.

Ex: If $m_{ij} = 1$: $s_i = s_j^{-1} = s_j$

If $m_{ij} = 2$: $s_i s_j s_i s_j = 1$

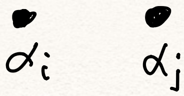
$$(s_i s_j s_i \underbrace{s_j) s_j} = s_j$$

$$(s_i s_j s_i) s_i = (s_j) s_i$$

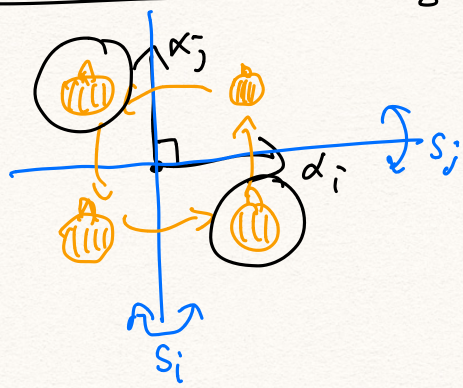
$$\boxed{s_i s_j = s_j s_i} \quad \text{(commute)}$$

If $m_{ij} > 2$: $\quad s_i s_j s_i s_j \cdots s_i s_j = 1$

Relations from Dynkin diagram:

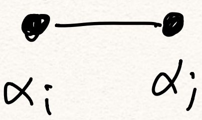


$\Rightarrow$



$$\boxed{s_i s_j s_i s_j = 1}$$

$m_{ij} = 2.$

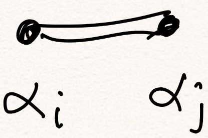


$s_i s_j s_i \underbrace{s_j s_i s_j} = 1$

$m_{ij} = 3$

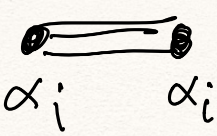
$$\boxed{s_i s_j s_i = s_j s_i s_j}$$

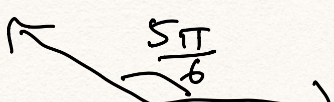
Braid relation.



$m_{ij} = 4$

$(s_i s_j)^4 = 1$





$m_{ij} = 6$

$(s_i s_j)^6 = 1$

(Composition of two reflections is a rotation)

Ex: Group generated by $|$ is $\mathbb{Z}$ $\subseteq \mathbb{Z}$

$\langle s, r \mid rs = 1 \rangle$ $\qquad$ $1, s, s^2, s^3, \ldots$

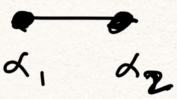
$\qquad\qquad\qquad\qquad\qquad$ $1, r, r^2, r^3, \ldots$

Lemma: Weyl groups are finite.

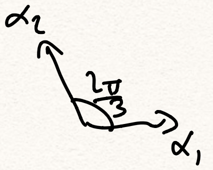
Pf: They are the group of symmetries of a finite set of vectors. (Orbit-Stabilizer).

$\square$

Ex: A_2:



$$\langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, \ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$



$$\cong$$

$$S_3$$

Model: $s_1 = (12)$ $\qquad$ $(12)(23)(12) = (13)$

$\qquad\quad$ $s_2 = (23)$ $\qquad$ $(23)(12)(23) = (13)$

Braid relation:

(13)

Def: A <u>reduced word</u> in a Weyl group is a sequence of generators $s_{i_1} s_{i_2} \cdots s_{i_r}$ s.t. if $w = s_{i_1} s_{i_2} \cdots s_{i_r}$, any product of generators that equals w has length at least r.

<u>i.e.</u> A reduced word for $w \in W$ is a minimal length factorization of w as a product of generators.

<u>Ex:</u> In $\langle s_1, s_2 \mid s_1^2 = s_2^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

Reduced words are:

$$\{ 1,\ s_1,\ s_2,\ s_1 s_2,\ s_2 s_1,\ \underbrace{s_1 s_2 s_1}_{\substack{\| \\ s_2 s_1 s_2}} \} = S_3$$

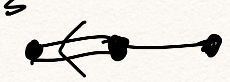
$$s_1 s_2 = \underbrace{s_1 \cancel{s_1 s_1} s_2}_{\text{not reduced}}$$

$s_2 s_1 s_2$ ↑ longest word ⇝ vertical symmetry of sl_3 crystals

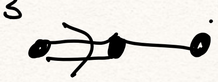
<u>Claim:</u> no other reduced words

- can't have $s_1 s_1$ or $s_2 s_2$
- $\underbrace{s_1 s_2 s_1 s_2} = s_2 s_1 \cancel{s_2 s_2} = \underline{s_2 s_1}$

<u>Ex:</u> B_3:



C_3:



$s_0 \quad s_1 \quad s_2$

$$W = \left\langle s_1, s_2, s_0 \;\middle|\; \begin{array}{l} s_1^2 = s_2^2 = s_0^2 = 1 \\ s_0 s_2 = s_2 s_0 \\ s_1 s_2 s_1 = s_2 s_1 s_2 \\ \underline{s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0} \end{array} \right\rangle$$

<u>Model:</u> $s_1 = (12)$

$s_2 = (2\,3)$

$s_0 =$ "negate first entry"

Def: The Hyperoctahedral group H_n is the group of permutations w in $S_{\{-n, -(n-1), \ldots, -1, 1, 2, \ldots, n\}}$

s.t. $w(-i) = -w(i)$ for all i

Writing elts of H_n: "signed permutation".

Ex: • $(1, -2, 3) \in H_3$ represents map

$(2\ -2)$

$1 \to 1 \qquad -1 \to -1$

$2 \to -2 \qquad -2 \to 2$

$3 \to 3 \qquad -3 \to -3$

• $3, -1, -2$ represents map in $S_{\{-3,-2,-1,1,2,3\}}$

$$\begin{pmatrix} 1 \to 3 & -1 \to -3 \\ 2 \to -1 & -2 \to 1 \\ 3 \to -2 & -3 \to 2 \end{pmatrix}$$

Cycle notation: $(1\ 3\ -2)(-1\ -3\ 2)$

• $(1\ 3\ -2\ -1\ -3\ 2)$ in cycle notation

As a signed permutation:

$$\begin{matrix} 3, & 1, & -2. \\ \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 \end{matrix}$$

$$\begin{array}{ccc} 1 & 2 & 3 \\ & \downarrow s_0 & \\ -1 & 2 & 3 \\ & \downarrow s_1 & \\ 2 & -1 & 3 \\ & \downarrow s_2 & \\ 2 & 3 & -1 \\ & \downarrow s_0 & \\ -2 & 3 & -1 \end{array}$$

$$\boxed{|H_n| = n! \cdot 2^n}$$

$$|H_3| = 3! \cdot 2^3 = 48$$

$$(2n)(2n-2)(2n-4)\cdots$$

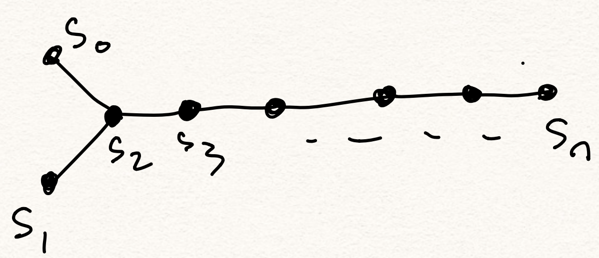
$$= 2^n (n)(n-1)\cdots$$

$$\underset{(2n)}{\underline{3}}\ \underset{(2n-2)}{\underline{\quad}}\ \underset{(2n-4)}{\underline{\quad}}\ \cdots$$

$$(2n)(2n-2)(2n-4)\cdots 2 = (2n)!!$$

$$2^n \cdot n(n-1)(n-2)\cdots 1 = 2^n \cdot n!$$

Type D:



$s_1 = (12)$

$W_{D_n} = \langle s_0, s_1, \ldots, s_n \mid$
- $s_i^2 = 1$
- $s_i s_j = s_j s_i$ for $|i-j| \geq 2$, $i, j \neq 0$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \neq 0$
- $s_0 s_j = s_j s_0$ for $j \neq 2$
- $s_0 s_2 s_0 = s_2 s_0 s_2 \rangle$

Interpretation: $W_{D_n} \subseteq W_{B_n} = H_n$

s_0: "swap and negate the first two entries"

$$\begin{array}{c} 1\ 2\ 3 \\ \downarrow s_0 \\ -2\ -1\ 3 \\ \downarrow s_1 \\ -1\ -2\ 3 \\ \downarrow s_0 \\ 2\ 1\ 3 \\ \downarrow s_1 \\ 1\ 2\ 3 \end{array}$$

$$\begin{array}{ccc} 1\ 2\ 3 & & 1\ 2\ 3 \\ \downarrow s_0 & & \downarrow s_2 \\ -2\ -1\ 3 & & 1\ 3\ 2 \\ \downarrow s_2 & & \downarrow s_0 \\ -2\ 3\ -1 & & -3\ -1\ 2 \\ \downarrow s_0 & & \downarrow s_2 \\ -3\ 2\ -1 & & -3\ 2\ -1 \end{array}$$

s_0 — s_2

$$\boxed{W_{D_n} = \{\text{signed permutations having an even \# of } - \text{ signs}\}}$$

$$|W_{D_n}| = n! \cdot \Big(\underbrace{\tbinom{n}{0} + \tbinom{n}{2} + \tbinom{n}{4} + \cdots}\Big)$$
$$= \boxed{n!\, 2^{n-1}}$$

$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots (-1)^n\binom{n}{n}$

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1 \end{array}$$

Proof that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$:

$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$

Therefore $\underbrace{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots}_{A} = \underbrace{\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots}_{B}$

$A = B, \quad A + B = 2^n \Rightarrow A = B = 2^{n-1}$

<u>Hwk:</u> Compute Weyl group of $G2$