Recall Abreu–Nigro thm: if $\chi : \mathbb{E}_n \to \mathbb{Q}(q)$ satisfies the <u>modular law</u> and

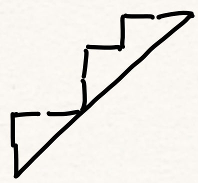
$$\chi(\underline{e}_\mu) = \begin{cases} \prod (\lambda_i)_q! & \lambda \text{ rearranges } \mu \\ 0 & \text{otherwise} \end{cases}$$

then $\chi(\underline{e}) = c_\lambda(\underline{e}; q)$ for any $\underline{e} \in \mathbb{E}_n$.

<u>Modular law:</u> for any $(\underline{e}, \underline{e}', \underline{e}'')$ in $\mathbb{E}_n$ satisfying either:

(i) $\exists\, \underset{[2,n]}{\overset{\uparrow}{i}}$ s.t. $e(i-1) < e(i) < e(i+1)$
$e(e(i)) = e(e(i)+1)$ $\boxed{\left(\text{where } e(n+1) = n-1\right)}$
and $e'(j) = e''(j) = e(j)$ $j \neq i$,
$e'(i) = e(i)+1$ $e''(i) = e(i)-1$

<u>Ex:</u> $\underline{e} = (0, \underline{0}, \underline{1}, \underline{3}, 3)$ $i=3$
$\underline{e}' = (0, 0, 2, 3, 3)$
$e'' = (0, 0, 0, 3, 3)$

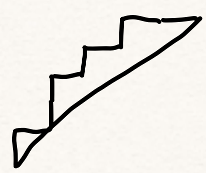


OR

(ii) $\exists\, i \in [n-1]$ s.t. $e(i+1) = e(i)+1$
and $e^{-1}(i) = \emptyset$

e', e'' match except at $i, i+1$, tie them

<u>Ex:</u> $e = (0, 0, 1, 2, 4)$ $i=3$
$e' = (0, 0, 2, 2, 4)$
$e'' = (0, 0, 1, 1, 4)$

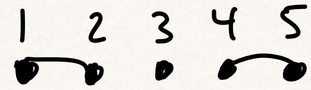


Then $\boxed{(1+q)\chi(\underline{e}) = q\,\chi(\underline{e}') + \chi(\underline{e}'')}$.

For 1st ex: graphs are

$(\underline{e})$

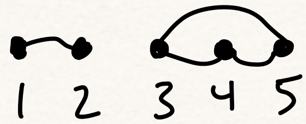
$(1+q)e_2\big((1+q+q^2)e_3+q\,e_{2,1}\big)$



$(\underline{e}')$

$(1+q)^2 e_{2,2,1}$

$(1+q)e_2 \cdot (3)_q!\, e_3$



$(\underline{e}'')$

For $\lambda=(3,2)$: $\quad \chi(\underline{e}) = c_{(3,2)}(\underline{e};q)$

$$\chi(\underline{e}) = (3)_q!$$
$$\chi(\underline{e}')=0$$
$$\chi(\underline{e}'')=(1+q)(3)_q! = (1+q)\chi(e) \quad ✓$$

For $\lambda=(2,2,1)$: $\quad \chi(\underline{e}) = c_{(2,2,1)}(\underline{e};q)$

$$\left.\begin{array}{l}\chi(\underline{e}) = q(1+q)\\ \chi(\underline{e}') = (1+q)(1+q)\end{array}\right\} \begin{array}{l}(1+q)\chi(e)\\ \quad = q\chi(e')\end{array}$$
$$\chi(\underline{e}'')=0$$

✓

<u>Note:</u> If $(\underline{e},\underline{e}',\underline{e}'')$ is a modular triple then $|e'|=|e|+1$, $|e''|=|e|-1$.

Thus if

$$p_\lambda(\underline{e};q) := q^{|\underline{e}|-|\underline{e}_\lambda|} \frac{c_\lambda(\underline{e};q)}{\prod (\lambda_i)_q!}$$

then since c_λ satisfies the modular law, we have

$$(1+q)p_\lambda(\underline{e};q) = q^{|\underline{e}|-|\underline{e}_\lambda|} \frac{(1+q)c_\lambda(\underline{e};q)}{\prod (\lambda_i)_q!}$$

$$= q^{|\underline{e}|-|\underline{e}_\lambda|} \frac{q c_\lambda(\underline{e}';q) + c_\lambda(\underline{e}'';q)}{\prod (\lambda_i)_q!}$$

$$= \left(q^{|\underline{e}'|-|e_\lambda|} \frac{c_\lambda(\underline{e}';q)}{\prod (\lambda_i)_q!}\right) + \left(q^{|\underline{e}''|-|e_\lambda|} \frac{q c_\lambda(\underline{e}'';q)}{\prod (\lambda_i)_q!}\right)$$

$$= p_\lambda(e';q) + q\, p_\lambda(e'';q).$$

<u>Conclusion:</u>

<u>Cor</u>: For $(\underline{e},\underline{e}',\underline{e}'')$ modular,

$$\boxed{(1+q)\,p_\lambda(\underline{e};q) = p_\lambda(\underline{e}';q) + q\, p_\lambda(e'';q).}$$

Note: this is the modular law with $q \mapsto q^{-1}$, so Abreu-Nigro also applies.

<u>Def</u>: $\chi_\lambda(\underline{e}) := \sum_{T \in SYT(\lambda)} P_T(\underline{e}; q)$, the prob. distribution

<u>Goal</u>: Check that χ_λ also satisfies

$$(1+q)\chi_\lambda(\underline{e}) = \chi_\lambda(\underline{e}') + q\chi_\lambda(\underline{e}'')$$

for any $(\underline{e}, \underline{e}', \underline{e}'')$ in Def 8 and

$$\chi_\lambda(\underline{e}_\mu) = \begin{cases} 1 & \lambda \text{ rearr. } \mu \\ 0 & \text{otherwise} \end{cases}$$

this is sufficient!

<u>Reformulation of def of χ_λ</u>:

<u>Def</u>: $V_n = \mathbb{Q}(q)$-vector space w/ basis $SYT(n)$

<u>Ex</u>: elt of V_3 is $q \begin{smallmatrix} 3 & \\ 1 & 2 \end{smallmatrix} + 2q^2 \begin{smallmatrix} 2 & \\ 1 & 3 \end{smallmatrix} - \begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix}$

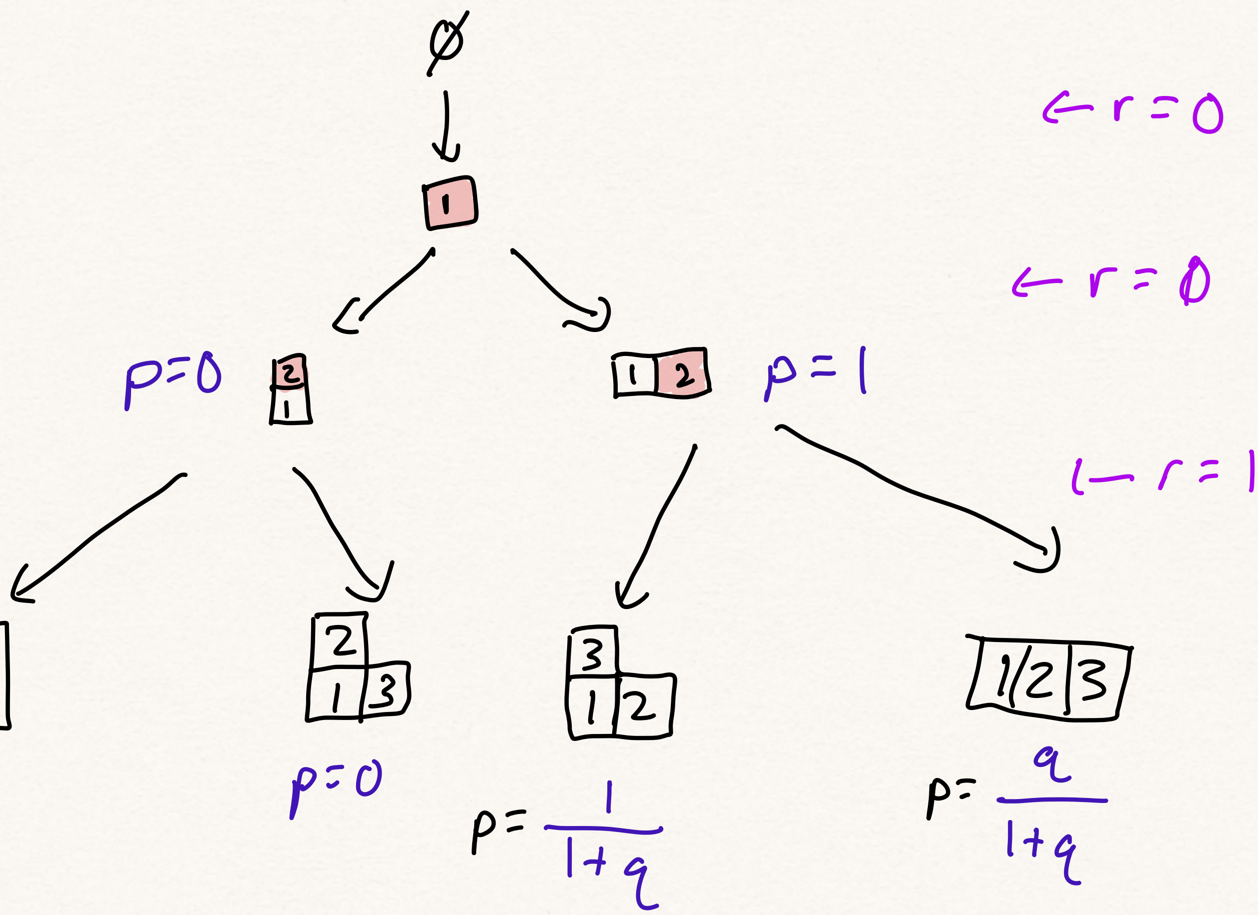
Note: $V_0 \cong \mathbb{Q}(q)$, spanned by empty tableau $\emptyset$

Def: For $n \in \mathbb{Z}_{\geq 0}$, $0 \leq r \leq n$, define linear map $\Omega_r : V_n \longrightarrow V_{n+1}$ by

$$\Omega_r(T) = \sum_{k=0}^{d^{(r)}(T)} \varphi_k^{(r)}(T;q)\, f_k^{(r)}(T)$$

for any $T \in SYT(n)$ (and extend by linearity).

Ex: Recall tree for $\underline{e} = (0,0,1)$:



So $\Omega_1\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\right) = \frac{1}{1+q} \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \frac{q}{1+q} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$

Def: Linear map $pr_\lambda : V_n \to \mathbb{Q}(q)$

by
$$pr_\lambda(T) = \begin{cases} 1 & T \in SYT(\lambda) \\ 0 & T \in SYT_n \setminus SYT(\lambda) \end{cases}$$

Def:

For $\underline{e} \in \mathbb{E}_n$, set
$$\Omega_{\underline{e}} = \Omega_{\underline{e}(n)} \circ \Omega_{e(n-1)} \circ \cdots \circ \Omega_{e(1)} : V_0 \to V_n$$

Then
$$\chi_\lambda(\underline{e}) = pr_\lambda(\Omega_{\underline{e}}(\emptyset))$$

Ex:
$$\chi_{(2,1)}((0,0,1)) = pr_{(2,1)}(\Omega_1 \circ \Omega_0 \circ \Omega_0(\emptyset))$$
$$= pr_{(2,1)}\left(\frac{1}{1+q} \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} + \frac{q}{1+q} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right)$$
$$= \frac{1}{1+q}$$

Notation: $[S:T] =$ coeff of T in $S \in V_n$

Ex:
$$\left[\Omega_{(0,0,1)}(\emptyset) : \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right] = \frac{q}{1+q}$$

Recall:

Goal: Check that χ_λ also satisfies

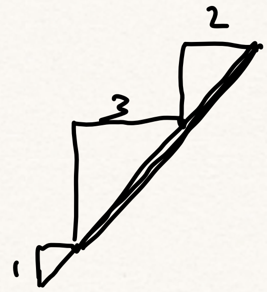
$$(1+q)\chi_\lambda(\underline{e}) = \chi_\lambda(\underline{e}') + q\chi_\lambda(\underline{e}'')$$

for any $(\underline{e}, \underline{e}', \underline{e}'')$ in Def 8 and

$$\chi_\lambda(\underline{e}_\mu) = \begin{cases} 1 & \lambda \text{ rearr. } \mu \\ 0 & \text{otherwise} \end{cases}$$

First check $\chi_\lambda(\underline{e}_\mu)$ computation:

Ex: $\underline{e}_{(2,3,1)} = (0,0,2,2,2,5)$



$$\Omega_5\Omega_2\Omega_2\Omega_2\Omega_0\Omega_0(\emptyset)$$

$$= \Omega_5\Omega_2^3\left(\boxed{1\,|\,2}\right)$$

$\tau_{r=2}$, no red boxes

$$= \Omega_5\Omega_2^2\left(\begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array}\right)$$

$\delta = (0,0,0)$
only can add to top

new boxes red

$$= \Omega_5\left(\begin{array}{|c|c|c|}\hline 3 & 4 & \\ \hline 1 & 2 & 5 \\ \hline\end{array}\right)$$

$\delta = (1,1,0,0,0)$ etc at each step

Note: $\Omega_0^n(\emptyset)$
$= \boxed{1\ 2 \cdots n}$
Since $r=0$ so all boxes are red

$$= \begin{array}{|lll}\hline 6 & & \\ 3 & 4 & \\ 1 & 2 & 5 \\ \hline \end{array}$$

So, tableau is formed by layering:

<u>Ex 2</u>: If $\mu = (2,4,5,1)$ then

$$\Omega_{e_\mu} = \begin{array}{lllll} 12 & & & & \\ 7 & 8 & & & \\ 3 & 4 & 9 & 10 & \\ 1 & 2 & 5 & 6 & 11 \end{array}$$

i.e. horizontal strips layered.

$$\Rightarrow \chi_\lambda(e_\mu) = \begin{cases} 1 & \lambda \text{ rearr. } \mu \\ 0 & \text{else} \end{cases}$$

QED.

<u>Now check modular law for (i):</u>

<u>Outline of steps:</u>

<u>Def</u>: $M_{m,n}$ = subspace of V_n spanned by

$$R_m(T) = T + q\frac{(L-1)_q}{(L+1)_q}\tau_m(T)$$

for $T \in SYT_n$ s.t.:

- box m is @ top of c'th col and $m+1$ is @ top of $(c+L)$th col
- $\tau_m(T)$ = exchange $\boxed{m} \leftrightarrow \boxed{m+1}$ (If $L=1$, $R_m(T)=T$)

<u>Ex:</u> $M_{3,5}$ spanned by:

- $R_3\left(\boxed{1\,2\,3\,4\,5}\right) = \boxed{1\,2\,3\,4\,5}$
- $R_3\left(\boxed{\begin{array}{llll} 5 & & & \\ 1 & 2 & 3 & 4 \end{array}}\right) = \boxed{\begin{array}{llll} 5 & & & \\ 1 & 2 & 3 & 4 \end{array}}$
- $R_3\left(\boxed{\begin{array}{lll} 3 & 5 & \\ 1 & 2 & 4 \end{array}}\right) = \boxed{\begin{array}{lll} 3 & 5 & \\ 1 & 2 & 4 \end{array}} + \frac{q}{(3)_q}\boxed{\begin{array}{lll} 4 & 5 & \\ 1 & 2 & 3 \end{array}}$
- $R_3\left(\begin{array}{llll} 3 & & & \\ 1 & 2 & 4 & 5 \end{array}\right) = \boxed{\begin{array}{llll} 3 & & & \\ 1 & 2 & 4 & 5 \end{array}} + \frac{q}{(3)_q}\boxed{\begin{array}{llll} 4 & & & \\ 1 & 2 & 3 & 5 \end{array}}$
- $R_3\left(\begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 4 & 5 \end{array}\right) = \boxed{\begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 4 & 5 \end{array}}$ (+ 0 · other)
- $R_3\left(\begin{array}{llll} 2 & & & \\ 1 & 3 & 4 & 5 \end{array}\right) = \boxed{\begin{array}{llll} 2 & & & \\ 1 & 3 & 4 & 5 \end{array}}$
- $R_3\left(\begin{array}{lll} 5 & & \\ 2 & & \\ 1 & 3 & 4 \end{array}\right) = \boxed{\begin{array}{lll} 5 & & \\ 2 & & \\ 1 & 3 & 4 \end{array}}$

Lemma: For any $T \in SYT_n$ and $0 \le r \le n$, we have $\Omega_r(\Omega_r(T)) \in M_{n+1,n+2}$

Ex: $T = \begin{matrix} 2 & 4 & \\ 1 & 3 & 5 \end{matrix}$, $r = 3$

$$\delta = (\underbrace{0}_{b_0=0}, \underbrace{1}_{a_1}, \underbrace{1, 1}_{b_1}, \underbrace{0, 0, 0}_{a_2})$$

$l = 1$

$$\Omega_r(T) = \varphi_0^3(T) \cdot \begin{matrix} 6 & & \\ 2 & 4 & \\ 1 & 3 & 5 \end{matrix} + \varphi_1^3(T) \cdot \begin{matrix} 2 & 4 & & \\ 1 & 3 & 5 & 6 \end{matrix}$$

$$= \frac{(1)_q}{(3)_q} \begin{matrix} 6 & & \\ 2 & 4 & \\ 1 & 3 & 5 \end{matrix} + \frac{q(2)_q}{(3)_q} \begin{matrix} 2 & 4 & & \\ 1 & 3 & 5 & 6 \end{matrix}$$

because

$$\boxed{\varphi_k(\delta) = q^{a_1 + \cdots + a_k} \prod_{i=1}^{k} \frac{(a_{i+1} + \cdots + a_k + b_i + \cdots + b_k)_q}{(a_i + \cdots + a_k + b_i + \cdots + b_k)_q} \prod_{i=k+1}^{l} \frac{(a_{k+1} + \cdots + a_i + b_{k+1} + \cdots + b_{i-1})_q}{(a_{k+1} + \cdots + a_i + b_{k+1} + \cdots + b_i)_q}}$$

So

$$\Omega_r(\Omega_r(T)) = \frac{1}{(3)_q} \Omega_3\left(\begin{matrix} 6 & & \\ 2 & 4 & \\ 1 & 3 & 5 \end{matrix}\right) + \frac{q(2)_q}{(3)_q} \Omega_3\left(\begin{matrix} 2 & 4 & & \\ 1 & 3 & 5 & 6 \end{matrix}\right)$$

$\delta = (0, 1, 1, 1, 0, 0, 0)$

$$= \frac{1}{(3)_q} \begin{array}{llll} 6 & & & \\ 2 & 4 & & \\ 1 & 3 & 5 & 7 \end{array} + \frac{q(2)_q}{(3)_q}\left(\frac{(1)_q}{(4)_q} \begin{array}{llll} 7 & & & \\ 2 & 4 & & \\ 1 & 3 & 5 & 6 \end{array} + \frac{q(3)_q}{(4)_q} \underset{\star}{\begin{array}{lllll} 2 & 4 & & & \\ 1 & 3 & 5 & 6 & 7 \end{array}} \right)$$

$\star$ is itself in $M_{6,7}$, and the other two are τ_6 switches with $L=3$ so the factor is $\frac{(2)_q}{(4)_q}$ and we're done.

$\square$

<u>Lemma:</u> $\Omega_r(M_{m,n}) \subseteq M_{m,n+1}$ for any $r<m$.

<u>Ex:</u> $\boxed{\begin{array}{llll} 3 & & & \\ 1 & 2 & 4 & 5 \end{array}} + \frac{q}{(3)_q} \boxed{\begin{array}{llll} 4 & & & \\ 1 & 2 & 3 & 5 \end{array}} \in M_{3,5}$

$(1,0,1,1,0,0) \quad \downarrow \Omega_2 \quad (\underbrace{1}_{b_0},0,1,\underbrace{1}_{b_1},\underbrace{0,0}_{a_2})$

Same coloring on T, $\tau_m(T)$, $m, m+1$ both red, coeffs preserved ✓

<u>Def:</u> $K_{m,n} \subseteq V_n$ linear span of

$T - \tau_m(T)$ for $T \in SYT(n)$
s.t. $\tau_m(T)$ well defined.

<u>Note:</u> $pr_\lambda(K_{m,n}) = 0$. (Kernel?)

<u>Lemma:</u> $\Omega_r(K_{m,n}) \subseteq K_{m,n+1}$ for any $r \neq m$.

<u>Pf:</u> $r \neq m$ means $m, m+1$ same color,
so T, $\tau_m(T)$ affected the
same way, still come in
swapped pairs $\square$

<u>Lemma:</u> $((1+q)\Omega_r - \Omega_{r+1} - q\Omega_{r-1})(M_{r,n}) \subseteq K_{r,n+1}$
for all r.
(maybe will work out an ex later)

<u>Putting it together:</u> For (e, e', e'') of
type (i), have $e(e(i)) = e(e(i)+1)$
so

(*) $\Omega_{e(e(i)+1)} \Omega_{e(e(i))} \cdots \Omega_{e(1)}(\emptyset)$

$$\in M_{e(i), e(i)+1}$$

Note: $e(i) < i$, so sequence $e(1), \ldots, e(i)$ extends $e(1), \ldots, e(e(i)+1)$. Also have by assumption $e(i-1) < e(i)$ so

$$\Omega_{e(i-1)} \Omega_{e(i-2)} \cdots \Omega_{e(1)}(\emptyset) \qquad (**)$$

at most applies Ω_r's w/ $r < e(i)$ to (*), so $(**) \in M_{e(i),i}$.

We also have $e(i) < e(i+1) \leq e(j)$ for $j \geq i+1$ so applying final lemma, have

$$(1+q)\Omega_{\underline{e}} - \Omega_{\underline{e}'} - q\Omega_{e''}(\emptyset) \in K_{e(i),n}.$$

So pr_λ of LHS is 0, which means χ_λ satisfies modular law.

$\square$

Similar lemmas for type (ii).