Going through Hikita's claimed proof of Stanley–Stembridge – Part I, overview/preliminaries

Recall:

Stanley–Stembridge Conjecture: The chromatic symmetric function $X_G(x)$ is e-positive for any unit interval graph G.

Intro/proof overview: Γ – unit interval graph.

Associate Hessenberg function

$$h = h(1), h(2), \ldots, h(n)$$

$h(i) =$ max neighbor of i in Γ (or i itself).

Define

$$\underline{e} = \underline{e}(1), \underline{e}(2), \ldots, \underline{e}(n)$$

$$\underline{e}(i) = n - h(n+1-i)$$

$$e(1) = n - h(n)$$

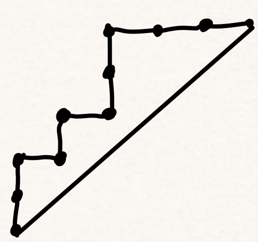
$$e(2) = n - h(n-1)$$

$$\vdots$$

$$e(n) = n - h(1)$$

- $\underline{e} : [n] \to \mathbb{Z}$
- $0 \leq \underline{e}(i) < i$
- $\underline{e}(i) \leq \underline{e}(i+1)$

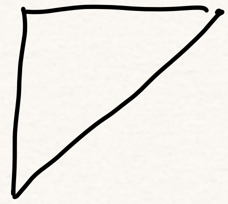
Ex:



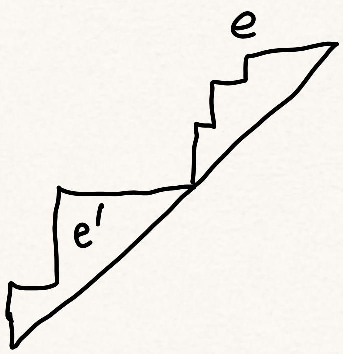
$$h = (2, 3, 5, 5, 5)$$

$$\underline{e} = (0, 0, 0, 2, 3)$$

Def: $\underline{e}_n = \underbrace{(0,0,0,\ldots,0)}_{n}$



Def: $\underline{e} \cup \underline{e}' = \underline{e}(1), \ldots, \underline{e}(n), n+\underline{e}'(1), \ldots, n+\underline{e}'(n')$



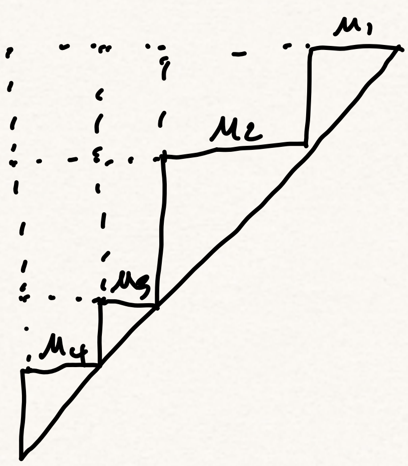
Def: $\underline{e}_\mu = \underline{e}_{\mu_1} \cup \cdots \cup \underline{e}_{\mu_\ell}$

$\mu = (\mu_1, \ldots, \mu_\ell)$ composition

Def: $|\underline{e}| = \sum \underline{e}(i)$

Lemma: $|e_\mu| = \sum_{i<j} \mu_i \mu_j$

Pf: $|e_\mu|$ is # squares above Dyck path:



Count each rectangle!

$\square$

<u>Def:</u> Write $\chi(\underline{e};q)$ for the chromatic quasisymmetric function for $\Gamma_{\underline{e}}$.

Write $$\chi(\underline{e};q) = \sum_{\lambda \vdash n} c_\lambda(\underline{e};q)\, e_\lambda .$$

<u>Previous work of Hikita:</u> ("in preparation"???)

$$\sum_{\lambda \vdash n} q^{|\underline{e}| - |\underline{e}_\lambda|} \frac{c_\lambda(\underline{e};q)}{\prod_i [\lambda_i]_q!} = 1$$

"easy application of the modular law", by Abreu-Nigro

Sounds like Hikita defines (q,t)-chromatic symmetrics in another forthcoming paper though.

<u>Ex:</u> $h=(2,3,3)$ $\quad \underline{e}=(0,0,1)$ $\quad \Gamma =$ 

$|\underline{e}|=1$

$$\chi(\underline{e};q) = (1+q+q^2)e_3 + q\, e_{(2,1)}$$

$c_{(3)} = 1+q+q^2 \qquad c_{(2,1)} = q \qquad c_{(1,1,1)} = 0$

LHS above has two terms:

$\underline{\lambda=(3)}$: $\quad q^{1-0} \frac{(1+q+q^2)}{(3)_q!} = \frac{q}{(1+q)}$

$\underline{\lambda=(2,1)}$: $\quad q^{1-2} \cdot \frac{q}{(1)_q!\,(2)_q!} = \frac{1}{1+q}$

$\Bigg\}$ sum is 1!

In class computation exercise

Def: $$p_\lambda(\underline{e};q) = q^{|\underline{e}| - |\underline{e}_\lambda|} \frac{c_\lambda(\underline{e};q)}{\prod [\lambda_i]_q!}$$

Shareshian-Wachs says $c_\lambda(\underline{e};q) \in \mathbb{N}[q]$, so for any $q \in \mathbb{R}_{>0}$, p_λ's are a probability distribution on the set of partitions λ of n (for a fixed $\underline{e}$).

- Hikita will explicitly construct these probabilities by induction on n
- This way, we don't need Shareshian-Wachs to know that they're positive at $q>0$.
- Plug in $q=1$ and it implies that c_λ is positive for the Stanley-Stembridge case!
- Note we still don't know Shareshian-Wachs from this proof strategy.

Goal: construct probability for $\underline{e}+(r) = (\underline{e}(1), \dots, \underline{e}(n), r)$ from that for $\underline{e}$.

Auxiliary probability distribution: $\{p_T(\underline{e};q) : T \in SYT(n)\}$

Given T, paint boxes larger than r red:

$$\begin{array}{|c|c|cc} \hline 4 & 5 & & \\ \hline 1 & 2 & 3 & \multicolumn{1}{|c|}{6} \\ \hline \end{array} \qquad r=4$$

Def: The color sequence $\delta^{(r)}(T) = (\delta_1, \ldots, \delta_{n+1})$

is $\delta_i = \begin{cases} 1 & \text{if top box of } i\text{-th col of } T \text{ is red} \\ 0 & \text{otherwise} \end{cases}$

Ex: $(0, 1, 0, 1, 0, 0, 0)$

Notation: a's and b's:

For any (T, r) $\exists!$ $\ell^r(T) = \ell$ ← (number) s.t.

$$\delta^r(T) = (1^{b_0}, 0^{a_1}, 1^{b_1}, 0^{a_2}, \ldots, 0^{a_\ell}, 1^{b_\ell}, 0^{a_{\ell+1}})$$

with $b_0 \geq 0$, $a_i, b_i > 0$ for $i > 0$.

Ex above: $(0^1, 1^1, 0^1, 1^1, 0^2)$

$b_0 = 0$ $b_1 = 1$ $b_2 = 1$

$a_1 = 1$ $a_2 = 1$ $a_3 = 2$

Def: For any $0 \leq k \leq \ell$, define

$$c_k^{(r)}(T) = \sum_{i=1}^{k} a_i + \sum_{i=0}^{k} b_i + 1$$

Ex: $k=0$ $c_0^{(r)}(T) = 1$

$k=1$ $c_1^{(r)}(T) = 3$

$k=2$ $c_2^{(r)}(T) = 5$

$\Bigg\}$ positions of columns to which we can add a box labeled $n+1$ to T!

Def: $f_k^{(r)}(T) =$ add $n+1$ to col $c_k^{(r)}(T)$.

Def: transition probability ~~from~~ from T to $f_k^{(r)}(T)$:

$$\varphi_k^{(r)}(T;q) = q^{\sum_{i=1}^{k} a_i} \prod_{i=1}^{k} \frac{\left[\sum_{j=i+1}^{k} a_j + \sum_{j=i}^{k} b_j\right]_q}{\left[\sum_{j=i}^{k} a_j + \sum_{j=i}^{k} b_j\right]_q} \prod_{i=k+1}^{l} \frac{\left[\sum_{j=k+1}^{i} a_j + \sum_{j=k+1}^{i-1} b_j\right]_q}{\left[\sum_{j=k+1}^{i} a_j + \sum_{j=k+1}^{i} b_j\right]_q}$$

Ex:

$$T \to \begin{matrix} 7 & & & \\ 4 & 5 & & \\ 1 & 2 & 3 & 6 \end{matrix} \qquad \varphi_0^{(4)}(T;q) = \frac{[3]_q}{[2]_q[4]_q}$$

$$T \to \begin{matrix} 4 & 5 & 7 & \\ 1 & 2 & 3 & 6 \end{matrix} \qquad \varphi_1^{(4)}(T;q) = \frac{q}{[2]_q^2}$$

$$T \to \begin{matrix} 4 & 5 & & & \\ 1 & 2 & 3 & 6 & 7 \end{matrix} \qquad \varphi_2^{(4)}(T;q) = \frac{q^2[3]_q}{[2]_q[4]_q}$$

Def: $p_\emptyset(\emptyset;q) = 1$, and

$$p_{T'}(\underline{e}+(r);q) = \begin{cases} \varphi_k^{(r)}(T)\,p_T(\underline{e};q) & \text{if } T' = f_k^{(r)}(T) \\ 0 & \text{otherwise} \end{cases}$$

In particular if T_i = delete all $j>i$ from T,

$$p_T(e;q) = \begin{cases} \prod_{i=1}^{n} \varphi_{k_i}^{(e(i))}(T_{i-1}) & \text{if } T_i = f_k^{(e(i))}(T_{i-1}) \ \forall i \\ 0 & \text{else} \end{cases}$$

<u>Note:</u> $p_T(\underline{e};q) \geq 0$ for $q \geq 0$. ✓

<u>Def:</u> $p_\lambda(\underline{e};q) = \sum_{T \in SYT(\lambda)} p_T(\underline{e};q)$.

<u>Thm:</u> The p_λ above matches the c_λ definition.

Lemmas about q-analogs

$$(n)_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

$$(n)_q! = (1)_q (2)_q \cdots (n)_q$$

<u>Lemma:</u> ① $(2)_q (m)_q = (m+1)_q + q(m-1)_q$

② $(m+1)_q (n)_q - (m)_q (n+1)_q = q^m (n-m)_q$

③ $(m+1)_q (n)_q - q(m)_q (n-1)_q = (m+n)_q$

<u>Pf:</u> ①: $(m)_q + q(m)_q = 1 + 2q + 2q^2 + 2q^3 + \cdots + 2q^{m-1} + q^m = (m+1)_q + q(m-1)_q$

etc.

<u>Def</u>: For $\delta = (1^{b_0}, 0^{a_1}, 1^{b_1}, 0^{a_2}, \dots, 0^{a_\ell}, 1^{b_\ell}, 0^{a_{\ell+1}})$

set

$$\varphi_k(\delta) = q^{a_1 + \dots + a_k} \prod_{i=1}^{k} \frac{(a_{i+1} + \dots + a_k + b_i + \dots + b_k)_q}{(a_i + \dots + a_k + b_i + \dots + b_k)_q} \prod_{i=k+1}^{\ell} \frac{(a_{k+1} + \dots + a_i + b_{k+1} + \dots + b_{i-1})_q}{(a_{k+1} + \dots + a_i + b_{k+1} + \dots + b_i)_q}$$

for $0 \le k \le \ell$. (go to ex below here)

Let c be s.t. c^{th} entry of δ is 1,

$$c = \sum_{j=1}^{m} a_j + \sum_{j=0}^{m-1} b_j + \beta$$

for some $0 \le m \le \ell$, $1 \le \beta \le b_m$.

Let δ' be formed by changing c^{th} entry to 0.

Define

$$c_k(\delta) = \sum_{j=1}^{k} a_j + \sum_{j=0}^{k} b_j + 1 \qquad \text{(first 0 of each block)}$$

<u>Lemma</u>: (i) If $1 < \beta < b_m$,

$$\varphi_k(\delta') = \begin{cases} \dfrac{(c - c_k(\delta) + 1)_q}{(c - c_k(\delta))_q} \cdot \varphi_k(\delta) & k < m \\[2ex] q\, \dfrac{(c_{k-1}(\delta) - c - 1)_q}{(c_k(\delta) - c)_q} \cdot \varphi_{k-1}(\delta) & k > m \end{cases}$$

(ii) If $\beta = 1$ and $b_m > 1$, we have

$$\varphi_k(\gamma') = \begin{cases} \dfrac{(c - c_k(\gamma)+1)_q}{(c - c_k(\gamma))_q} \cdot \varphi_k(\gamma) & \text{if } 0 \le k < m \\ q\,\dfrac{(c_k(\gamma)-c-1)_q}{(c_k(\gamma)-c)_q} \cdot \varphi_k(\gamma) & \text{if } m \le k \le \ell \end{cases}$$

(iii) If $\beta = b_m$ and $b_m > 1$,

$$\varphi_k(\gamma') = \begin{cases} \text{same as} & \text{\_\_\_\_\_\_} \\ \text{part (ii)} & \\ & m < k \le \ell. \end{cases}$$

(iv) If $b_m = 1$,

$$\varphi_k(\gamma') = \begin{cases} \dfrac{(c - c_k(\gamma)+1)_q}{(c - c_k(\gamma))_q}\, \varphi_k(\gamma) & 0 \le k < m \\ q\,\dfrac{(c_{k+1}(\gamma)-c-1)_q}{(c_{k+1}(\gamma)-c)_q}\, \varphi_{k+1}(\gamma) & m \le k \le \ell - 1 \end{cases}$$

<u>Ex:</u> $T = \begin{array}{|llllllll} 9 & 10 & & & & & & \\ 3 & 4 & 8 & 13 & & & & \\ 1 & 2 & 5 & 6 & 7 & 11 & 12 & 14 \end{array}$ $\qquad r = 8$

$$\gamma = (\underbrace{1,1}_{b_0},\underbrace{0}_{a_1},\underbrace{1}_{b_1},\underbrace{0}_{a_2},\underbrace{1,1,1}_{b_2},\underbrace{0,0,0,0,0,0,0}_{a_3})$$

$$\varphi_0(\gamma) = \frac{(a_1)_q}{(a_1+b_1)_q} \cdot \frac{(a_1+a_2+b_1)_q}{(a_1+a_2+b_1+b_2)_q} = \frac{1}{(2)_q}\frac{(3)_q}{(6)_q}$$

$$\varphi_1(8) = q\,\frac{(b_1)_q}{(a_1+b_1)_q}\cdot\frac{(a_2)_q}{(a_2+b_2)_q} = q\cdot\frac{1}{(2)_q}\cdot\frac{1}{(4)_q}$$

$$\varphi_2(8) = q^2\,\frac{(a_2+b_1+b_2)_q}{(a_1+a_2+b_1+b_2)_q}\,\frac{(b_2)_q}{(a_2+b_2)_q} = q^2\frac{(5)_q}{(6)_q}\,\frac{(3)_q}{(4)_q}$$

Note: $\varphi_0(8)+\varphi_1(8)+\varphi_2(8)$

$$= \frac{(3)_q(4)_q + q(6)_q + q^2(2)_q(5)_q(3)_q}{(2)_q(4)_q(6)_q}$$

$$= \frac{q\cancel{(2)_q}(3)_q + \cancel{(1+q)}(6)_q + q^2\cancel{(2)_q}(5)_q(3)_q}{\cancel{(2)_q}(4)_q(6)_q}$$

$$= \frac{(q+q^2(5)_q)(3)_q + (6)_q}{(4)_q(6)_q} = \frac{q(3)_q+1}{(4)_q} = 1$$

$$\boxed{\begin{aligned} &(2)_q(n)_q = (n+1)_q + q(n-1)_q \\ &(m+1)_q(n)_q - (m)_q(n+1)_q = q^m(n-m)_q \\ &(m+1)_q(n)_q - q(m)_q(n-1)_q = (m+n)_q \end{aligned}}$$

$$(3)_q(4)_q = q(2)_q(3)_q + (6)_q$$

<u>Lem:</u> $\sum_k \varphi_k(8) = 1$ in general.

$$c_0(\delta) = b_0 + 1 \qquad\qquad = 3$$
$$c_1(\delta) = b_0 + a_1 + b_1 + 1 \qquad = 5$$
$$c_2(\delta) = b_0 + a_1 + b_1 + a_2 + b_2 + 1 = 9$$

} cols we can add a box to

Ex: $c = 7$

$$\delta' = (\underbrace{\overset{b_0}{1}, \overset{a_1}{1}, \overset{b_1}{0}, \overset{a_2}{1}, \overset{\beta}{0}}, \underset{b_2}{1}, \underset{a_3}{\underline{0}}, \underset{b_3}{1}, \underbrace{0, 0, 0, 0, 0, 0, 0}_{a_4})$$

$$m = 2, \qquad \beta = 2$$

$k=1$:

$$\varphi_1(\delta') = q \frac{(b_1)_q}{(a_1+b_1)_q} \cdot \frac{(a_2)_q}{(a_2+b_2)_q} \cdot \frac{(a_2+b_2+a_3)_q}{(a_2+b_2+a_3+b_3)_q}$$

$$= q \cdot \frac{1}{(2)_q} \cdot \frac{1}{(2)_q} \cdot \frac{(3)_q}{(4)_q}$$

whereas

$$\varphi_1(\delta) = q \frac{1}{(2)_q} \frac{1}{(4)_q}$$

differs by $\dfrac{(3)_q}{(2)_q} = \dfrac{(c - c_k(\delta) + 1)_q}{(c - c_k(\delta))_q}$

$$= \frac{(7-5+1)_q}{(7-5)_q} \qquad \checkmark$$

Modular law: $\mathbb{E}_n = \{\underline{e} \text{ functions}\}$

Def: Say $\chi : \mathbb{E}_n \to \mathbb{Q}(q)$ satisfies the modular law if for any (e, e', e'') in $\mathbb{E}_n^3$ satisfying either

(i) $\exists\, i \in \{2, \ldots, n\}$ s.t. $\underline{e}(i-1) < \underline{e}(i) < \underline{e}(i+1)$ and $e(e(i)) = e(e(i)+1)$, and s.t. $e'(j) = e''(j) = e(j)$ for $j \neq i$, $e'(i) = e(i)+1$, $e''(i) = e(i)-1$

OR

(ii) $\exists\, i \in [n-1]$ s.t. $e(i+1) = e(i)+1$, $e^{-1}(i) \neq \emptyset$, $e'(j) = e''(j) = e(j)$ for $j \neq i, i+1$, $e'(i) = e'(i+1) = e(i+1)$, $e''(i) = e''(i+1) = e(i)$

then $\boxed{(1+q)\chi(e) = q\chi(e') + \chi(e'')}$

$\boxed{\underline{\text{Note}}: e(n+1) = n-1}$

Thm (Abreu–Nigro) $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$. If χ satisfies modular law and

$$\chi(e_\mu) = \begin{cases} \prod (\lambda_i)_q! & \lambda \text{ rearr. } \mu, \\ 0 & \text{else} \end{cases}$$

Then $\chi(\underline{e}) = c_\lambda(\underline{e}; q) \quad \forall \underline{e}$.