Math 601: Advanced Combinatorics I
Homework 7 - Due Dec 6

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 15. The maximum possible score on this homework is 15 points. See the syllabus for details.

Problems

1. Show that the action of S_n on tableaux crystals by the reflection operators s_i defined in class is a well-defined group operation (replacing the unpaired string $i^a(i+1)^b$ with $i^b(i+1)^a$), via the following steps (you may hand in a subset of the following problems).

   (a) (2 points) (Warmup example.) Compute $s_2(T)$ where T is the tableau below:

3	4								
2	2	2	3	3					
1	1	1	1	2	2	2	2	3	3

   (b) (2 points) Explain why it suffices to show that $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ when acting on tableaux, for all i.

   (c) (2 points) Show that, using the compatibility of JDT slides with crystal operators (which you may use as a fact), it suffices to show the previous statement holds for $i = 1$, and therefore it suffices to work with $\mathfrak{sl}_3$ crystals, that is, tableaux whose letters are all $1, 2, 3$.

   (d) (2 points) Show that one can further reduce to the case that the tableau shape has two rows.

   (e) (2 points) Show that, using the symmetry in the relation $(s_1s_2)^3 = 1$, it suffices to consider the case when the tableau has partition weight, that is, its content is (a, b, c) for some $a \geq b \geq c$.

   (f) (2 points) Show that the reading word of the tableau must be of the form $2^d3^e1^a2^f3^g$ where $d + f = b$ and $e + g = c$.

   (g) (5 points) Use casework on the relative values of d, e, a, f, g to show that the relation $s_1s_2s_1 = s_2s_1s_2$ always holds when acting on the reading word in the previous part.

2. (3 points) Compute the chromatic symmetric function of the triangle graph, that is, the complete graph K_3, and express it in terms of elementary symmetric functions and in terms of Schur functions.

3. (3 points) Show that the q-degree of the Hall-Littlewood polynomial $\widetilde{H}_\mu(x;q)$ is equal to $\sum \binom{\mu_i'}{2}$ where μ_i' are the parts of the conjugate partition, in other words, the lengths of the columns of μ. Show that this is attained when the tableau is the highest weight tableau of content and shape μ.

4. (4 points) Show directly that the dimension of the Springer fiber $\mathcal{B}_\mu$ is equal to $\sum \binom{\mu_i'}{2}$ as defined in the previous problem.