Math 601: Advanced Combinatorics I
Homework 4 - Due Oct 11

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 15. The maximum possible score on this homework is 15 points. See the syllabus for details.

Problems

NOTE: Some problems below concern $\mathfrak{sl}_2$ and others concern $\mathfrak{sl}_3$. Read carefully!

1. (5 points) Describe a "ballot-type" condition for a word of 1's and 2's to be *lowest weight* for $\mathfrak{sl}_2$, that is, that F sends the word to 0. Prove that your condition is correct. Do the same for $\mathfrak{sl}_3$ and the two lowering operators.

2. (3 points) In the $\mathfrak{sl}_3$ tableau crystals discussed in class, describe the *lowest weight* tableau of shape λ. Conclude that every irreducible $\mathfrak{sl}_3$ representation has a unique lowest weight.

3. (2 points) Write the element E_{12} in $\mathfrak{sl}_3$ as an 8×8 matrix in the adjoint representation, by computing how E_{12} acts on the basis $\{E_{ij} | i \neq j\} \cup \{H_{12}, H_{23}\}$ defined in class, via the adjoint operator $[E_{12}, -]$.

4. (2 points) Show that the embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3$ that sends

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   is indeed an injective homomorphism of Lie algebras.

5. (2 points) Compute $V_{1,1} \otimes V_{1,1}$ as a direct sum of irreducible representations of $\mathfrak{sl}_3$, by interpreting each in terms of two types of $\mathfrak{sl}_2$ chains and using the L method.

6. (2 points) Compute $V_{1,1} \otimes V_{1,1}$ as a direct sum of irreducible representations of $\mathfrak{sl}_3$ using the Littlewood-Richardson rule. What Schur function product does this correspond to, and with how many variables?

7. Recall that the *complete homogeneous symmetric function* $h_\mu(x_1, \ldots, x_n)$, for a partition $\mu = (\mu_1, \ldots, \mu_k)$, can be defined as the product $h_{\mu_1} \cdots h_{\mu_k}$ where h_d is the sum of all monomials in $x_1, \ldots, x_n$ of degree d.

   (a) (1 point) Which Schur function in n variables is $h_d(x_1, \ldots, x_n)$ equal to?

   (b) (2 points) Show that $h_\mu(x_1, x_2, x_3)$ is the character of the tensor product of the k irreducible $\mathfrak{sl}_3$ representations

$$V^{(\mu_1,0)} \otimes \cdots \otimes V^{(\mu_k,0)}$$

   (where $V^{(a,b)}$ denotes the irreducible representation with highest weight (a, b)).

   (c) (3 points) Recall from Math 502 the definition of the Kostka numbers $K_{\lambda\mu}$ and state the definition here. Also recall the formula

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$

   What happens to this formula when we only restrict this identity to three variables x_1, x_2, x_3? What does this mean about the tensor product in part (b)?