Math 601: Advanced Combinatorics I
Homework 3 - Due Sep. 27

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 15. The maximum possible score on this homework is 15 points. See the syllabus for details.

Problems

1. **Writing $\mathfrak{sl}_2(\mathbb{C})$ representations in terms of matrices:**

   (a) (1 point) Draw the four-dimensional irreducible representation V^3 of $\mathfrak{sl}_2(\mathbb{C})$ as a graph with edges labeled by E, F, H.

   (b) (3 points) Consider the basis $\{v_3, v_1, v_{-1}, v_{-3}\}$ of V^3 given by the eigenvectors of H with weights $3, 1, -1, -3$ respectively. Write the images of E, F, H in $\mathfrak{gl}(V^3)$ as 4×4 matrices with respect to this basis.

   (c) (1 point) If these 4×4 matrices are e, f, h, confirm that $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$.

2. (1 points) Write the tensor product $V^4 \otimes V^3$ as a direct sum of irreducible representations of $\mathfrak{sl}_2$.

3. (3 points) Write the tensor product $V^2 \otimes V^2 \otimes V^2$ as a direct sum of irreducible representations of $\mathfrak{sl}_2$.

4. (5 points) Let $V = V^n$ and $W = W^m$ be irreducible $\mathfrak{sl}_2$ representations and let $v_n, v_{n-2}, \ldots$ and $w_m, w_{m-2}, \ldots$ be the eigenvectors of H with weights according to their subscripts. Find a full set of explicit highest weight vectors of $V \otimes W$, and deduce the Clebsch-Gordan rule explicity from your construction.

   For instance, we can check that $v_n \otimes w_m$ is highest weight by applying E and showing we get 0. Similarly one can check that $v_{n-2} \otimes w_m - v_n \otimes w_{m-2}$ is highest weight. Find the remaining highest weight vectors explicitly.

5. (3 points) A **highest weight vector** of a representation of $\mathfrak{sl}_2$ is a vector v such that $Ev = 0$. Let v be a vector in $V_1^{\otimes n}$, written as a word w of 1's and 2's. Show that v is a highest weight vector if and only if the word $w = w_1 \cdots w_n$ has the *ballot* (also called the *lattice* or *Yamanouchi* property) that for all $i \leq n$, the suffix $w_i w_{i+1} \cdots w_n$ has at least as many 1's as 2's.

6. (5 points) Two words p, q of length n in letters 1 and 2 are *Knuth equivalent* to each other if and only if p can be obtained from q by a sequence of **elementary Knuth moves**, defined as follows. Given three consecutive letters of the form 212, it can be changed to 221 (or vice versa) in the middle of the word. Alternatively consecutive letters 121 can be changed to 211 and vice versa.

   Prove that the $\mathfrak{sl}_2$ crystal operators commute with Knuth equivalence.