Math 601: Advanced Combinatorics I
Homework 2 - Due Sep. 13

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 15. The maximum possible score on this homework is 15 points. See the syllabus for details.

Problems

1. (1 point) Let $\lambda = (5,4,1)$, and $V \cong \mathbb{C}^4$ with elementary basis vectors
$$e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad ,e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad ,e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$
Write the following element of $S^\lambda V$ as a Young tableau:
$$(e_1 \wedge e_3 \wedge e_4) \otimes (e_1 \wedge e_3) \otimes (e_2 \wedge e_4) \otimes (e_2 \wedge e_4) \otimes e_4$$

2. (3 points) Suppose $V = \mathbb{C}^3$. Compute the dimension of the irreducible GL_3 representation $V^{(k,k)}$ in terms of the parameter k.

3. (3 points) Write the tableau
$$\begin{array}{|c|c|}\hline 5 & 6 \\ \hline 4 & 3 \\ \hline 1 & 2 \\ \hline\end{array}$$
in terms of the semistandard Young tableaux basis in the corresponding Schur module, using a sequence of column exchange relations.

4. (4 points) Prove Sylvester's Lemma, which was a step towards understanding the Schur modules as discussed in class. That is, show that for $n \times n$ matrices M, N, we have
$$\det(M)\det(N) = \sum \det(M')\det(N')$$
where we choose any k columns of N, and the sum ranges over all ways of interchanging k columns of M with the chosen ones in N, to form matrices M' and N'.

5. (3 points) Prove that Sylvester's Lemma in the proof we outlined in class is indeed equivalent to the column exchange relations for the determinants D_T. In particular, given two columns with entries $c_1, \ldots, c_r$ and $d_1, \ldots, d_s$ (with the d column to the right), the two matrices in question should be M with $M_{i,j} = z_{i,c_j}$ for $1 \le i, j \le r$ and N with $N_{i,j} = z_{i,d_j}$ for $1 \le i \le r$ and $1 \le j \le s$, and $N_{i,j} = 0$ for $s < j \le r$ and $i \ne j$, $N_{i,j} = 1$ for $s < j \le r$ and $i = j$.

6. (2 points) Show that the commutator bracket $[X,Y] = XY - YX$ is a Lie bracket, by showing that it is antisymmetric and satisfies the Jacobi identity
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

7. Using the formal infinitesimal ϵ satisfying $\epsilon^2 = 0$ as described in class, describe the elements of the Lie algebra corresponding to each of the following Lie groups.
   (a) (1 point) $\mathrm{SO}_n(\mathbb{C})$
   (b) (1 point) $\mathrm{Sp}_{2n}(\mathbb{C})$
   (c) (1 point) The torus $T_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ of diagonal invertible matrices
   (d) (1 point) The Borel subgroup $B_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ of upper triangular invertible matrices

8. (2 points) Show, using the ϵ method, that the Lie algebra of the orthogonal group $O_n(\mathbb{C})$ is isomorphic to that of the special orthogonal group $\mathrm{SO}_n(\mathbb{C})$. Why does this not contradict the bijective correspondence between connected Lie groups and Lie algebras?