Math 601: Advanced Combinatorics I
Cumulative Homework B - Due Dec 6

For this homework, all problems must be handed in to receive full credit. The problems are worth a total of 30 points. No collaboration is allowed, and it should be treated as a mini-take-home midterm/review.

Problems

1. (3 points) Draw the $\mathfrak{sl}_3$ crystal for weight $(3, 3, 0)$.

2. (4 points) Prove that the elements of the hyperoctahedral group, written in cycle notation as a permutation on $\{\pm 1, \dots, \pm n\}$, has all of its cycles coming in either pairs of the form $(a_1 \ \cdots \ a_k)(-a_1 \ \cdots \ -a_k)$, or of the form $(a_1 \ \cdots \ a_k \ -a_1 \ -a_2, \dots, -a_k)$.

3. (5 points) Define the Lie algebra $\mathfrak{so}_{2n+1}$ as $\{X : X^T S + SX = 0\}$ where

$$S = \begin{pmatrix} 1 & & \\ & & I_n \\ & I_n & \end{pmatrix}$$

   where I_n is the $n \times n$ identity matrix and 1 is in the upper left corner. Write down what an arbitrary element X looks like, and using the fact that with respect to this setup the torus is simply the set of diagonal matrices X satisfying these conditions, explain how one obtains the type B root system.

4. (2 points) What is the dimension of the adjoint representation of $\mathfrak{so}_7$?

5. (4 points) Explain why the set of 5th roots of unity in the plane does not form a root system. Which axioms of root systems does it satisfy?

6. (3 points) Compute the evacuation of the Young tableau below, and then evacuate again, and show you have returned to the starting tableau.

5			
2	7	8	
1	3	4	6

7. (3 points) Compute the Hall-Littlewood polynomial $\widetilde{H}_{(2,1,1)}(x; q)$.

8. (4 points) Let $w = w_1 \cdots w_n$ is a word of partition content, and suppose $w_1 \neq 1$. Let $w' = w_2 \cdots w_n w_1$ be formed by cycling w_1 around to the end of the word. Show that $\mathrm{cc}(w') = \mathrm{cc}(w) - 1$ where cc is cocharge. This operation is called *cyclage*.

9. (2 points) Give a counterexample showing that the formula in the above problem does not hold in general when $w_1 = 1$.