Math 601: Advanced Combinatorics I
Cumulative Homework A - Due Oct 4

For this homework, all problems must be handed in to receive full credit. The problems are worth a total of 30 points. No collaboration is allowed, and it should be treated as a mini-take-home midterm/review.

Problems

1. (4 points) Consider the dihedral group D_4 (the symmetry group of a square, including reflections and rotations) acting on the plane by the corresponding reflection and rotation matrices. Show that this representation of D_4 is an irreducible 2-dimensional representation over both $\mathbb{R}$ and $\mathbb{C}$.

2. (3 points) Prove that if G is a finite abelian group, then all of its irreducible representations are 1-dimensional. (Hint: you may want to look up some resources showing that commuting matrices are simultaneously diagonalizable.)

3. (5 points) Suppose A is an $m \times m$ matrix and B is an $n \times n$ matrix. Recall that the **tensor product** of the matrices A and B is the matrix having block form

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}$$

Show that, if $\rho : G \to \mathrm{GL}(\mathbb{C}^m)$ and $\sigma : G \to \mathrm{GL}(\mathbb{C}^n)$ are two representations of a group G (thought of as collections of matrices $\rho(g)$ and $\sigma(g)$), then the tensor product of these two representations, defined as the action of G on $\mathbb{C}^m \otimes \mathbb{C}^n$ by $g(v \otimes w) = (gv) \otimes (gw)$, is the map

$$\rho \otimes \sigma : G \to \mathrm{GL}(\mathbb{C}^{mn})$$

given by

$$\rho \otimes \sigma(g) = \rho(g) \otimes \sigma(g).$$

4. (3 points) Recall the RSK bijection from Math 502, and use an appropriate version of it to give a combinatorial proof that the dimensions of the spaces on both sides of the Schur-Weyl duality formula

$$\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = \bigoplus_{\lambda} V_\lambda \otimes V^\lambda$$

are equal. (Recall that the left hand side is a tensor product of k copies of $\mathbb{C}^n$, and the right hand side is summing over all λ of size k with at most n rows.)

(Hint: You do not need to know the actual RSK algorithm to do this problem, only what objects it forms bijections between. This can be found in Stanley chapter 7, for instance.)

5. (2 points) Use the ϵ method to find the Lie algebra associated to the Lie group $B_n \subseteq \mathrm{SL}_n(\mathbb{C})$ of upper triangular matrices in SL_n.

6. (3 points) Let V_i be the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight i. Compute the decomposition of $V_3 \otimes V_5$ into irreducibles.

7. (a) (5 points) Show that the total number of ballot sequences of 1's and 2's of length $2n$ is $\binom{2n}{n}$, and that the number of ballot sequences of 1's and 2's of length $2n+1$ is $\binom{2n+1}{n+1}$.
   (b) (2 points) If V_1 is the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ having highest weight 1, what does this imply about the decompositions of $V_1^{\otimes 2n}$ and $V_1^{\otimes 2n+1}$ into irreducibles?

8. (3 points) Starting with the ballot word 12211122121111, draw the corresponding $\mathfrak{sl}_2$ chain by applying the lowering operator F until it is no longer possible.