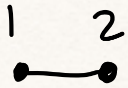
Def: Chromatic symmetric fn of a graph G w/ vertex set $\{1,\dots,n\}$, is

$$X_G(\underline{x}) = \sum_{\substack{\kappa : G \to \mathbb{Z} \\ \text{proper}}} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

Ex: $G =$ 

Colors from $\mathbb{Z}$.

Proper colorings: no two vertices have same color

Colorings: 

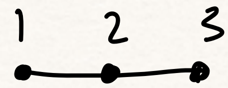
Monomial: $x_1 x_2$ $\quad x_2 x_1$ $\quad x_1 x_3$ $\quad x_3 x_1$ $\quad x_2 x_3 \quad \dots$

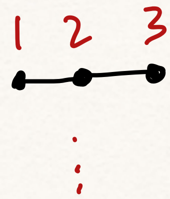
$$\begin{aligned} X_G &= 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + \cdots \\ &= 2e_2 \end{aligned}$$

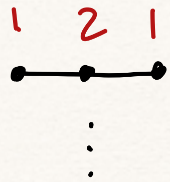
Lemma: X_G symmetric.

Pf: Given a proper coloring w/ monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if we apply a permutation π in $S_{\mathbb{Z}}$ to all the colorings we get a monomial $x_{\pi(1)}^{\alpha_1} \cdots x_{\pi(n)}^{\alpha_n}$. Thus the coefficients only depend on the partition formed by the exponents, and so X_G is symmetric.

$\square$

Ex: $X_G(x)$ for $G =$ 



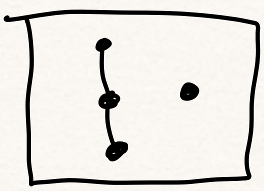


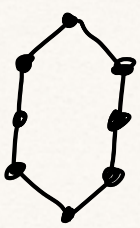
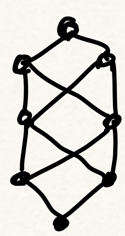
$6x_1x_2x_3 + \cdots$ $\quad$ $x_1^2x_2 + \cdots$

$$= 6m_{111} + m_{21}$$
$$= 6e_3 + e_1e_2 - 3e_3$$
$$= 3e_3 + e_1e_2$$

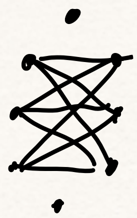
Stanley–Stembridge Conjecture: For nice enough G, $X_G(\underline{x})$ is e-positive: it expands positively in the elementary basis, w/ pos. integer coeffs.

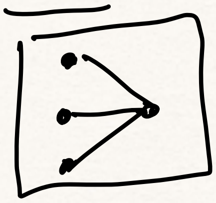
Nice enough: Incomparability graphs of 3+1-free posets (claw-free incomparability graphs).

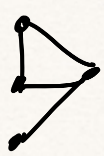
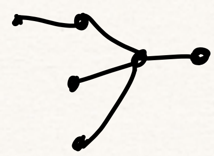
Def: A poset P is (3+1)-free if no induced subposet is: 

Ex:  NOT.  IS.

<u>Incomparability Graph</u>: G_P = • vertex set P
• Edge btwn v, w if v, w incomp. in P ($v \not\leq w$, $w \not\leq v$).

<u>Ex</u>:  

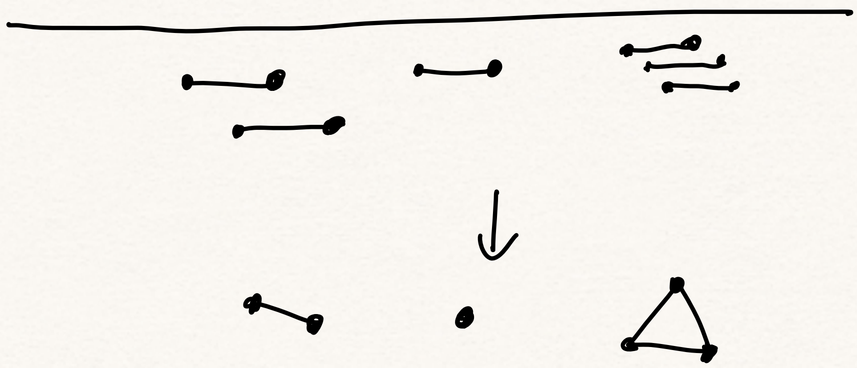
<u>Def</u>: A <u>claw</u> is an induced subgraph isomorphic to . We say G is <u>claw-free</u> if it has no claw.

<u>Ex</u>:  is claw-free!  is not.

<u>Thm</u>: (Guay-Paquet) Suffices to show it holds for <u>unit interval graphs</u>:

- Vertices are unit intervals in $\mathbb{R}$
- Edge between $v, w \iff$ the intervals overlap.

Ex:



Exercise: Why are unit interval graphs claw-free?

Known: Schur positivity (Gasharov).

Chromatic Quasisymmetric fns: An ascent of a proper coloring κ is a pair of vertices i, j with $i<j$ and $\kappa(i) < \kappa(j)$.

Write $\operatorname{asc}(\kappa)$ for # ascents. Then define

$$X_G(x;q) = \sum_{\substack{\kappa:[n]\to\mathbb{Z} \\ \text{proper}}} q^{\operatorname{asc}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

Ex:

$1\ 2\ 3 \to q^2$

$1\ 3\ 2 \to q$

$2\ 3\ 1 \to q$

$2\ 1\ 3 \to q$

$3\ 1\ 2 \to q$

$3\ 2\ 1 \to 1$

$1\ 2\ 1 \to q$

$2\ 1\ 2 \to q$

$$X_G(x;q) = (1+4q+q^2)e_3 + qM_{2,1}$$
$$= \boxed{(1+q+q^2)e_3 + qe_{2,1}}$$

Shareshian-Wachs Conjecture: for unit interval graphs G,

$X_G(x;q)$ is symmetric and

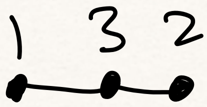
e-positive: coefficients in e basis are in $\mathbb{N}[q]$. (Polynomials in q w/ positive integer coefficients, i.e. they q-count something!)

Remark: $X_G(\underline{x};q=1) = X_G(\underline{x})$;

it's a q-analog.

Ex:

$X_G(x;q)$ not always symmetric, same underlying graph can have different labelings:



1 2 3

1 3 2

⋮

$121 \to q^2$

$212 \to 1$

$$\boxed{q^2 M_{2,1} + M_{1,2} + (1+4q+q^2)e_3}$$

Def: A quasisymmetric function is a bounded degree sum of monomials in $x_1, x_2, \ldots$ such that the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ matches that of

$$x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \quad \text{for any}$$

$$i_1 < i_2 < \cdots < i_k .$$

In other words, invariant under order-preserving maps (shifts).

Def: Monomial quasisymmetric function M_α (composition α) is the minimal Qsym fn that contains $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ as a term.

$$M_\alpha = \sum_{1 \le i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

Note: $M_{2,1} + M_{1,2} = m_{2,1}$ – can "symmetrize"

Lemma:
- Sum of two qsym fns is qsym.
- Product of two qsym fns is qsym.

Cor: $\mathrm{QSym}_{\mathbb{Z}}(\underline{x})$ forms a ring!

$$\Lambda_{\mathbb{Z}}(x) \subseteq \mathrm{QSym}_{\mathbb{Z}}(\underline{x}).$$

Connection to Hessenberg varieties

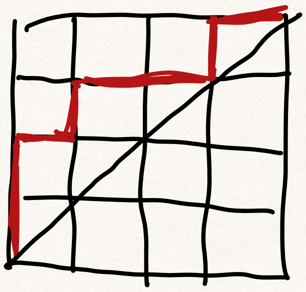
Shareshian-Wachs (proved by Brosnan-Chow) also noticed, for G unit interval,

$$\omega X_G(x;q) = \mathrm{Frob}_q(H^*(\mathrm{Hess}(M, h_G)))$$

"Hessenberg variety"

Def: A **Hessenberg function** $h: [n] \to [n]$ is a function s.t. $h(i) \geq i$ for all i, h weakly increasing.

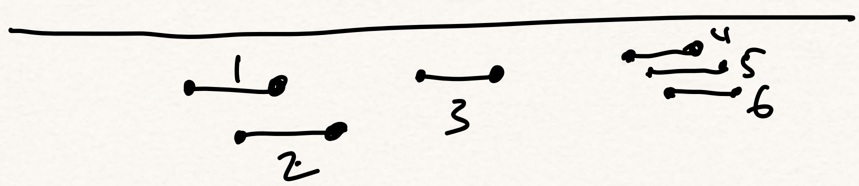
Alt: h is heights of horiz. steps of a Dyck path:

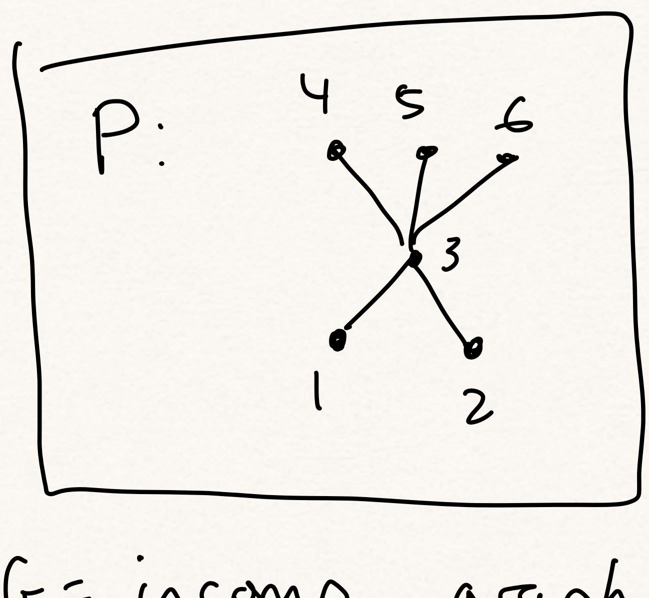


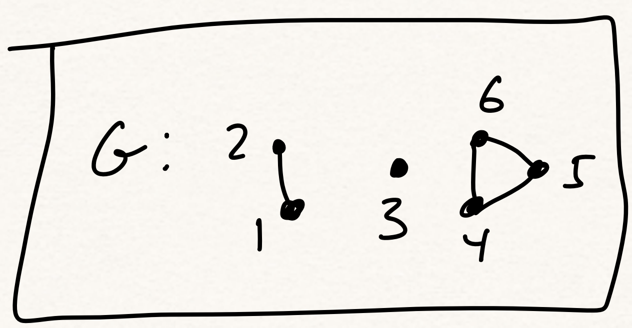
$h = (2,3,3,4)$ in vector form:
$h(1)=2$, $h(2)=3$,
$h(3)=3$, $h(4)=4$.

Graphs $\to$ Hessenbergs: Unit interval graph $G = (V,E)$:

$\rightsquigarrow$ Poset P on vertex set V with $v \leq w$ if unit interval v completely to left of unit interval w.



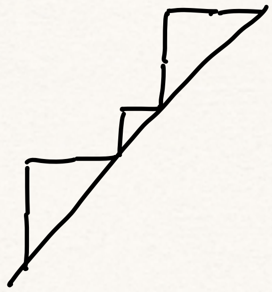




Then G = incomp. graph of P.

$P \to h$: define $h(i) =$ largest j s.t. $j \not> i$ in P

$h(1) = 2$
$h(2) = 2$
$h(3) = 3$
$h(4) = 6$
$h(5) = 6$



Bijection: can go back from $h \to P$ by defining $i <_P j$ iff $j \in \{h(i)+1, h(i)+2, \dots n\}$.

Summary: From unit interval graph G, can make Hessenberg fn h_G.

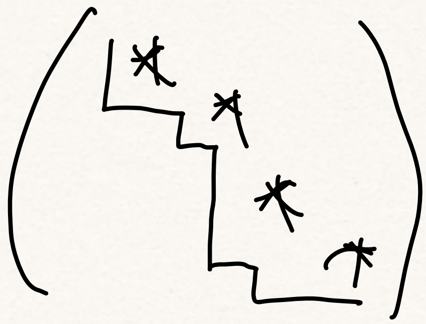
Def: M is a regular semisimple $n \times n$ matrix if it is diagonalizable w/ n distinct eigenvalues.

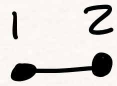
$$M = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$ for instance.

Def: The Hessenberg variety $\mathrm{Hess}(M, h)$ is

$$\{ V_\bullet \in Fl_n \mid M V_i \subseteq V_{h(i)} \ \forall i \}.$$

Motivation: Hessenberg spaces ("almost upper triangular")

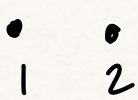


Ex: Path graph $G =$ 

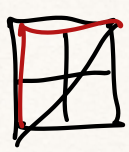
$$X_G(x;q) = (1+q) e_2$$

$$\omega X_G(x;q) = (1+q) h_2$$

$$= (1+q) s_{\square\square}$$

Poset $P =$ 

$h = (2,2)$

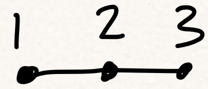


$$\mathrm{Hess}(M,h) = \{ V_1 \subseteq \mathbb{C}^2 \mid MV_1 \subseteq \mathbb{C}^2 \}$$
$$= Fl_2 = \mathbb{P}^1$$

$$H^*(\mathbb{P}^1) = \mathbb{C}[x]/(x^2)$$

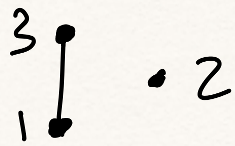
basis $1, x$

trivial trivial. ✓

Ex: $G =$ 

$$X_G(x;q) = (1+q+q^2)\, e_3 + q\, e_{2,1}$$

$$\omega X_G(x;q) = (1+q+q^2)\, h_3 + q\, h_2 h_1$$
$$= (1+q+q^2)\, s_{3} + q(s_{3} + s_{21})$$
$$= (1+2q+q^2)\, s_{3} + q\, s_{21}$$

$P =$ 

$h = (2,3,3)$

$$\mathrm{Hess}(M,h) = \{ V_1 \subseteq V_2 \subseteq \mathbb{C}^3 : MV_1 \subseteq V_2 \}$$

Tymoczko: GKM graphs, dot action for S_n action on $H^*(\mathrm{Hess}(M,h))$.