Math 601 (Advanced Combinatorics) Lecture Notes

Maria Gillespie

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1 Introduction

There is a growing body of knowledge that may be considered to be “classical” combinatorics. This involves permutations and combinations, bijections, recurrence and generating functions, graph theory, algorithms, and set systems such as matroids and combinatorial designs.

These basic combinatorial objects and tools have now been established as being useful throughout mathematics and the sciences. However, many of these objects were not always so ‘established’. The theory of combinatorial designs, for instance, was considered to be a purely recreational form of mathematics stemming from Latin squares and other riddles, until it rose to prominence in the 1900’s as its applications to agricultural science experiments became apparent. At that point the study of designs exploded and became a mainstream area of combinatorics.

This leads to an interesting philosophical question: how do we know when a combinatorics problem is ‘important’ to investigate? On the one hand, it is important to take fun problems about Latin squares and extend them for the sake of building fun combinatorial theory; indeed, down the road an important application might arise. On the other hand, it is important to look at problems from other fields of mathematics or science with a combinatorial mindset, to figure out what the current important problems are and what combinatorial tools need to be developed in order to solve them.

To summarize, there are two types of ‘importance’ that may apply to a combinatorial problem or theory:

1. It is a natural extension of previously solved problems in an established area of combinatorics.

2. It arises from an important question in a different area of math or science.

In fact, these two types are closely intertwined, and pursuing both are necessary to make new discoveries. Graph theory, for instance, may have arisen first in studies of maps, in which roads and bridges connect various towns or landmarks (type 2 above). Later, Euler and others studied the resulting natural questions about graph theory in a theoretical manner (type 1). This theory was discovered to have applications to countless other fields of study, such as computer science, social networks, neuroscience, and more (type 2). These then lead to more natural theoretical questions about graphs (type 1), and the theory becomes even stronger for any future applications that may arise.

As this is a second year graduate course in combinatorics, the goal of this class is to demonstrate how one modern area of combinatorics rose to prominence due to having type 2 importance. Specifically, we will focus on the area of algebraic combinatorics that is sometimes called **combinatorial representation theory**, as it first arose from important questions in representation theory and particle physics.

The combinatorial tools that arose in this field - namely, Young tableaux, symmetric functions, crystals, and reflection groups - have since proven extremely useful to many other areas of study, including intersection theory in algebraic geometry, polytope theory, and probabilistic systems such as particle exclusion processes. Combinatorial representation theory is therefore currently rising to greater prominence in mathematics, and perhaps will itself eventually be considered to be part of "classical" combinatorics.

In order to introduce this rich area of combinatorics, we will first focus on some representation theoretic background from a high level view, and then build the combinatorial tools needed to work with representations in a discrete manner.

2 A brief introduction to representation theory

We start with a brief overview and motivation of representation theory. It will be example-heavy and many proofs will be omitted, since the focus of this class will be on the relevant combinatorics.

We will primarily be studying representations of three objects: groups, Lie groups, and Lie algebras. We start with groups, since they are the most well-known and the simplest to define.

2.1 Representations of groups

A **group** is a set G along with an identity element $e \in G$ and a multiplication $* : G \times G \to G$ satisfying:

- **Associativity**: $(a * b) * c = a * (b * c)$ for any $a, b, c \in G$
- **Identity**: $e * a = a * e = a$ for any $a \in G$
- **Inverses**: For any $a \in G$, there exists a unique $b \in G$ such that $a * b = b * a = e$.

We often write the multiplication as concatenation, for instance writing ab instead of $a * b$, as a shorthand.

Abstract groups can be difficult to get an intuitive grasp on or work with directly. For instance, here are two different group structures on the set $\{e, a, b, c\}$, written out as full multiplication tables.

G:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

H:

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

There are more useful ways of **representing** the elements of these groups that allow us to get a better handle on them. One might notice that the first group, G, is a copy of $C_2 \times C_2$, whereas the second, H, is C_4. We can therefore represent them as symmetry groups of diagrams in the plane.

Indeed, $C_2 \times C_2$ is the symmetry group of a non-circular ellipse, or in general any shape that is fixed by both a horizontal and vertical reflection but not by any other reflection or rotation (other than rotation by 180 degrees, which is the composition of the two reflections). So for instance we can think of $C_2 \times C_2$ as the symmetry group of the set of four points $\{(2, 0), (-2, 0), (0, 1), (0, -1)\}$. If we label them as in Figure 1 at left, then we can represent the group elements as the permutations

$$e = \mathrm{id},\ a = (13),\ b = (24),\ c = (13)(24).$$

This is called a **permutation representation** of the group G.

2.1.1 Examples of representations

We can alternatively think of the reflections as transformations of the plane, leading to a **matrix representation**.

Example 2.1. Since a is the vertical reflection, it may be represented as the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which sends a point (x, y) (written as a column vector) to its reflection $(-x, y)$ about the y-axis. Similarly we have

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Multiplying two of these matrices corresponds to composing the transformations, so we have represented the group elements as matrices in a way that matrix multiplication captures the structure of the group.

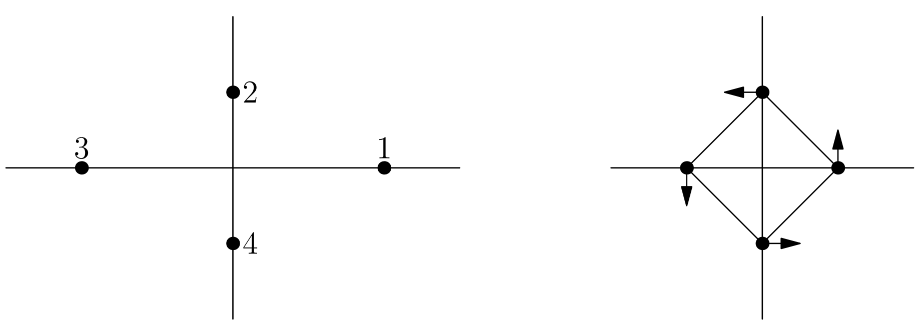

Figure 1: Diagrams having symmetry groups G and H respectively.

Example 2.2. In the case of the cyclic group H, the rotation matrices by $0°, 90°, 180°, 270°$ respectively are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that this representation is a little more cumbersome to work with than the previous example, because the matrices are not all diagonal. In general we want to find representations that are "as diagonal as possible". Indeed, if there is some matrix representation in which every group element is represented by a "block diagonal" matrix of the form

$$\begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & 0 \\ & & \ddots & \\ 0 & & & \boxed{A_k} \end{pmatrix}$$

where $A_1, A_2, \ldots, A_k$ are square matrices of fixed sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$, then multiplying two such matrices simply boils down to multiplying each of the block components together. In fact, each such block gives rise to a smaller matrix representation of the group consisting of $\lambda_i \times \lambda_i$ submatrices of the original matrices. For instance, the two sub-representations that arise from Example 2.1 are

$$e \to (1),\ a \to (-1),\ b \to (1),\ c \to (-1)$$

and

$$e \to (1),\ a \to (1),\ b \to (-1),\ c \to (-1).$$

Some elements are represented by the same matrix, but they are still representations because every element is represented by a matrix and multiplication of matrices corresponds to the group multiplication. Representations in which each element corresponds to a different matrix are called **faithful**. The two examples above are not faithful representations.

Let us now look back at Example 2.2. Is there some change of coordinates we can perform so that all the matrices are diagonal? In other words, are they **simultaneously diagonalizable**, meaning that there exists a single matrix P such that for all elements M of our set of matrices, PMP^{-1} is diagonal? Here's where the following theorem comes in handy.

Theorem 2.3. *A set of matrices is simultaneously diagonalizable if and only if they are each diagonalizable and all commute with each other.*

In this case, since we are representing abelian groups, the matrices must all commute, so if they are diagonalizable then they are simultaneously diagonalizable. And to check if a matrix is diagonalizable, one needs to see if it has an orthogonal set of eigenvectors.

Alas, over $\mathbb{R}$, the rotation matrices have no eigenvectors. But over $\mathbb{C}$, it is a different story!

Indeed, interpreting it as a *complex* matrix representation, we have that all four matrices have the common eigenvectors $(1, i)$ and $(i, 1)$. With respect to this basis, the four matrices e, a, b, c act as the diagonal matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

and indeed we have a "more diagonal" representation.

2.1.2 Formal definitions

We now define representations of groups in three different ways. As we saw above, the characteristics of a representation depend on what field you are working over (say $\mathbb{R}$ vs $\mathbb{C}$) and so we define them for a fixed arbitrary field $\mathbb{F}$. Throughout the course, if $\mathbb{F}$ is unspecified, we may assume that it is $\mathbb{C}$.

Definition 2.4 (Definition 1)**.** A **representation** of a group G over a field $\mathbb{F}$ is a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{F})$$

where $\mathrm{GL}_n(\mathbb{F})$ is the group of invertible $n \times n$ matrices over $\mathbb{F}$.

Definition 2.5 (Definition 2)**.** A **representation** of a group G over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space V along with an action $G \curvearrowright V$ by linear transformations, i.e. a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Definition 2.6 (Definition 3)**.** A **representation** of a group G over a field $\mathbb{F}$ is an $\mathbb{F}G$-module V. (Here $\mathbb{F}G$ is the **group ring** consisting of formal linear combinations of elements of G over $\mathbb{F}$. A **module** is essentially a "vector space over a ring".)

Exercise 2.7. (Essential!) Prove that all three definitions are equivalent.

Exercise 2.8. For the examples of the groups G and H from Examples 2.1 and 2.2, express these representations as a vector space with an action, and as a module.

With these definitions, we can then express the notions of diagonalizability and block form more cleanly in terms of direct sums and irreducible representations.

Definition 2.9. The **direct sum** of two representations V and W of a group G (thought of as vector spaces with an action of G) is the space $V \oplus W$ along with the action $G \curvearrowright V \oplus W$ by $g \cdot (v, w) = (g \cdot v, g \cdot w)$.

Exercise 2.10. Show that if all the matrices in a matrix representation of a group can be conjugated by the same change of basis matrix to be written in block form

$$\begin{pmatrix} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \ddots & \\ 0 & & & \boxed{A_k} \end{pmatrix}$$

where $A_1, A_2, \ldots, A_k$ are square matrices of fixed sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$, then the entire representation can be written as a direct sum of k smaller matrix representations corresponding to the k blocks.

In a direct sum $V \oplus W$, both V and W are **G-invariant subspaces** or **sub-representations** of $V \oplus W$. In general, any subspace of a representation V (thought of as a vector space) that is fixed by the action of G is called a **sub-representation** of the action of G on V.

Definition 2.11. A representation is **irreducible** if it has no proper sub-representations.

Note a representation V being irreducible is not in general equivalent to being **indecomposable**, meaning that it cannot be written as a direct sum of two nonzero sub-representations.

Exercise 2.12. Consider the representation of the group

$$B = \{\text{upper triangular matrices in } \mathrm{GL}_2(\mathbb{C})\}$$

given by its defining action on $\mathbb{C}^2$, that is, every matrix is represented by itself. Show that it has a one-dimensional sub-representation, but that it does not decompose as a direct sum of irreducibles.

However, the notions of irreducibility and indecomposability are equivalent for **finite** groups (and for simple Lie groups and semisimple Lie algebras, as we will see later.)

Theorem 2.13. *For a finite group G, a representation V of G is irreducible if and only if it is indecomposable.*

Exercise 2.14. Consider the matrix representation of S_3 as the symmetry group of the triangle with coordinates $(1, 0), (-1/2, \sqrt{3}/2), (-1/2, -\sqrt{3}/2)$ in the plane. Show that this is an irreducible 2-dimensional representation, even over $\mathbb{C}$.

Maschke's theorem says further that (for finite groups G) if V has a sub-representation W, then $V = W \oplus U$ for some U.

Corollary 2.15. *Let V be a representation of a finite group G. Then V can be decomposed as a direct sum of irreducible representations.*

In fact, this decomposition is **unique** (up to isomorphisms of the irreducible representations). This can be proven using **Schur's Lemma**, which is a statement about *homomorphisms* of representations. A **homomorphism** $f : V \to W$ is simply a homomorphism as $\mathbb{F}G$-modules. Schur's lemma can be stated as follows.

Lemma 2.16. *If V and W are irreducible representations of G and $f : V \to W$ is a homomorphism, then f is either the 0 map or an isomorphism.*

Exercise 2.17. Consider the representation of S_3 in which each permutation $\pi \in S_3$ is sent to its corresponding permutation matrix P, in which $P_{i,j} = \begin{cases} 1 & j = \pi(i) \\ 0 & j \neq \pi(i) \end{cases}$.

1. Find a common eigenvector of all of the permutation matrices.
2. Write the representation as a direct sum of irreducible representations.

2.1.3 Tensor products and characters

In addition to direct sum, the **tensor product** is another fundamental operation on representations.

Definition 2.18. Given vector spaces V and W over a field $\mathbb{F}$, the **tensor product** of V and W is the vector space $V \otimes W$ defined by

$$V \otimes W = \mathbb{F}\langle v \otimes w : v \in V, w \in W \rangle / Q$$

where Q is the sub-vector space generated by all relations of the form

$$(av_1 + bv_2) \otimes (cw_1 + dw_2) - ac(v_1 \otimes w_1) - ad(v_1 \otimes w_2) - bc(v_2 \otimes w_1) - bd(v_2 \otimes w_2)$$

where $a, b, c, d \in \mathbb{F}$ and $v_1, v_2 \in V, w_1, w_2 \in W$. (Note that $\mathbb{F}\langle v \otimes w : v \in V, w \in W \rangle$ denotes the vector space of all formal linear combinations of the symbols $v \otimes w$ where v and w are any vectors in v and w).

Exercise 2.19. Prove that, if $v_1, \ldots, v_n$ form a basis for V and $w_1, \ldots, w_m$ form a basis for W, then $\{v_i \otimes w_j : i \leq n, j \leq m\}$ forms a basis for $V \otimes W$, and therefore $\dim V \otimes W = nm$.

We can now use this definition to define the tensor product of two representations.

Definition 2.20. If V and W are representations of the group G, then $V \otimes W$ along with the "diagonal action" of G given by $g(v \otimes w) = (gv \otimes gw)$ is the **tensor product** of V and W.

In matrix terms, suppose A is an $m \times m$ matrix and B is an $n \times n$ matrix. Then the **tensor product** of the matrices A and B is the matrix having block form

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}$$

Exercise 2.21. Show that, if $\rho : G \to \mathrm{GL}(\mathbb{C}^m)$ and $\sigma : G \to \mathrm{GL}(\mathbb{C}^n)$ are two representations of G (thought of as collections of matrices $\rho(g)$ and $\sigma(g)$), then the tensor product of these two representations is the map

$$\rho \otimes \sigma : \mathrm{GL}(\mathbb{C}^{mn})$$

given by

$$\rho \otimes \sigma(g) = \rho(g) \otimes \sigma(g).$$

Exercise 2.22. (Trivial representation acts as multiplicative identity) Let $V_0 = \mathbb{C}$ be the trivial representation of the group G, in which every element of G acts as the identity (of size 1). Show that, for any representation W of G, we have $V_0 \otimes W \cong W$.

Exercise 2.23. Show that tensor product distributes over direct sum:

$$V \otimes (W \oplus U) = (V \otimes W) \oplus (V \otimes U)$$

and that tensor product is symmetric:

$$V \otimes W \cong W \otimes V.$$

Tensor products and direct sums both play nicely with *characters*, defined as follows.

Definition 2.24. The **character** of a representation $\rho : G \to \mathrm{GL}(V)$ is the map $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \mathrm{tr}(\rho(g))$.

We will learn much more about characters later in the course, but for now the essential facts are as follows.

Theorem 2.25. *The character of a representation uniquely determines the representation. Moreover, characters are additive on direct sums and multiplicative on tensor products.*

2.2 Lie groups

Now that we have defined the basic concepts of representation theory in the context of finite groups, we turn to Lie groups and Lie algebras, for which the story is somewhat more complicated but very analogous.

Definition 2.26. A (real or complex) **Lie group** is a real smooth manifold G (over $\mathbb{R}$ or $\mathbb{C}$ respectively) along with a group structure whose group multiplication

$$G \times G \to G$$

and inverse map $G \to G$ are differentiable. Equivalently, we must have that the map

$$G \times G \to G(x, y) \mapsto xy^{-1}$$

is differentiable.

We will not recall the precise definition of manifold here, but it will not be necessary for moving forward in combinatorial representation theory. Roughly speaking, a *manifold* is a topological space that is locally Euclidean; around each point there is an open set isomorphic to $\mathbb{R}^n$ (or $\mathbb{C}^n$) for some fixed n. In this case it is called an n*-dimensional* real (or complex) manifold.

Remark 2.27. Those with a more geometric mindset might visualize a smooth n-dimensional real variety instead, and indeed, the representation theory of Lie groups is nearly identical to that of *algebraic groups*, which are algebraic varieties along with a group structure whose multiplication and inverse maps are regular maps. All of the examples of Lie groups that we will be considering throughout the class will also be algebraic groups.

We now present many examples of Lie groups and methods of constructing new Lie groups from old.

2.2.1 First examples

Example 2.28. The space $\mathbb{R}^n$ is an n-dimensional real Lie group under vector addition. Likewise, $\mathbb{C}^n$ is an n-dimensional complex Lie group, and a $2n$-dimensional real Lie group.

Example 2.29. The space $\mathbb{R} - \{0\}$ is a 1-dimensional real Lie group under multiplication, and $\mathbb{C}^* = \mathbb{C} - \{0\}$ is a complex Lie group under multiplication.

Example 2.30. The groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ under matrix multiplication are real and complex Lie groups respectively, of dimension n^2. Topologically, GL_n (over either $\mathbb{R}$ or $\mathbb{C}$) can be realized as the subspace of the space of $n \times n$ matrices M_n (under the Euclidean metric using the entries of the matrices as coordinates) that avoids the hypersurface defined by the equation $\det(A) = 0$.

The example of GL_n above is crucial, as many more Lie groups can be constructed as closed subgroups of GL_n. Such a Lie group is called a **matrix Lie group**. In particular, the following fact will allow us to easily define most matrix Lie groups:

Proposition 2.31. *Any closed subgroup of a Lie group is a Lie group.*

Since the topological structure is sometimes hard to get a handle on, it is often useful to use the above proposition along with the following fact about closed subvarieties of algebraic groups such as GL_n.

Proposition 2.32. *Let G be a matrix Lie group. If $H \subset G$ is the solution set in G to a system of finitely many polynomial equations in the entries of the matrices in G, then H is closed in G.*

In fact, the above proposition is true for any algebraic variety, but here we will only use it for matrix Lie groups.

Let us now use Propositions 2.31 and 2.32 to construct many more examples of matrix Lie groups.

Example 2.33. Define T_n to be the subgroup of diagonal invertible matrices in GL_n (over either $\mathbb{R}$ or $\mathbb{C}$). Then T_n is a closed subgroup since it is defined by setting all of the off-diagonal entries to 0, all of which are polynomial equations (of degree 1). Thus T_n is a Lie group.

The Lie group T_n is called the **maximal torus** in GL_n, due to the fact that $T_2(\mathbb{C}^*)^2$, which is topologically a torus, and no larger torus is a subgroup of GL_n.

In general, given a Lie group G, its **maximal torus** is defined as the maximal compact, connected, abelian subgroup T of G.

Example 2.34. Define B_n to be the subgroup of upper-triangular matrices in GL_n. Then B_n is a closed subgroup since it is defined by setting all of the entries strictly below the diagonal to 0, and multiplying two upper-triangular matrices yields another upper-triangular matrix.

The Lie group B_n is called the (canonical) **Borel subgroup** in GL_n. In general a **Borel subgroup** of a matrix Lie group G is a maximal connected solvable subgroup. It is known that all Borel subgroups of G are conjugate to one another, so in particular they can be obtained by conjugating the subgroup B_n of upper triangular matrices by some change of basis matrix.

The Borel in GL_n can also be thought of as the stabilizer of the complete flag $0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \langle e_1, e_2, \ldots, e_n \rangle = \mathbb{F}^n$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ and where e_i is the i-th standard basis vector. In general any Borel in GL_n is the stabilizer of a complete flag.

Example 2.35. A **parabolic subgroup** is a proper subgroup containing the Borel. In GL_n, the parabolic subgroups are the stabilizers of *partial flags*. The stabilizer of the flag $0 = V_0 \subseteq V_{\lambda_1} \subseteq V_{\lambda_1+\lambda_2} \subseteq \cdots \subseteq V_{\lambda_1+\cdots+\lambda_k} = \mathbb{F}^n$ is the set of all block upper triangular matrices of the form

$$\begin{pmatrix} \boxed{A_1} & & * & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & 0 & & \boxed{A_k} \end{pmatrix}$$

where $A_1, \ldots, A_k$ are invertible square matrices of sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$. For a partition λ, this is called the parabolic subgroup P_λ.

Exercise 2.36. Compute the dimension of the parabolic subgroup P_λ over its field of coefficients.

Example 2.37. The subgroup of the Borel B_n in which all of the diagonal entries are 1 is called the **unipotent subgroup** N_n. The **unipotent parabolic** N_λ is the subgroup of P_λ in which the diagonal block matrices A_i are the identity matrix of size λ_i for all i.

Exercise 2.38. Compute the dimension of the unipotent parabolic N_λ over its coefficient field.

2.2.2 The classical groups

Example 2.39. The groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$, known as the **special linear groups**, are the subgroups of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ respectively consisting of matrices A with $\det(A) = 1$. Since $\det(A)$ is a polynomial in the matrix entries, we know it SL_n a closed subgroup and therefore is a Lie group.

The special linear group can also be thought as the group of volume-preserving linear transformations.

The next two examples are defined as subgroups of GL_n that fix a certain bilinear form.

Example 2.40. The **special orthogonal group** $\mathrm{SO}_n(\mathbb{R})$ is the group of matrices of determinant 1 that preserve a fixed symmetric, positive definite bilinear form $\langle , \rangle$ on $\mathbb{R}^n$. Over $\mathbb{C}$, the special orthogonal group $\mathrm{SO}_n(\mathbb{C})$ is the stabilizer of a fixed nondegenerate symmetric bilinar form.

The canonical example of such a bilinear form (in both cases of $\mathbb{R}$ and $\mathbb{C}$) is the "dot product" in which

$$\langle v, w\rangle = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = v^T w,$$

and any other symmetric, positive definite bilinear form yields a symmetry group that is conjugate to the copy of SO_n coming from the canonical choice.

The condition that a matrix A preserves the form is equivalent to saying that $\langle Av, Aw\rangle = \langle v, w\rangle$ for all v, w. But then $\langle Av, Aw\rangle = (Av)^T(Aw) = v^T A^T A w$, and this equals $v^T w$ for all v and w, so the condition is equivalent to $A^T A = I$, or $A^{-1} = A^T$. This, along with the condition $\det(A) = 1$, gives a set of polynomials that define the special orthogonal group as a closed subgroup of GL_n.

Geometrically, the group $\mathrm{SO}_n(\mathbb{R})$ can also be thought of as the group of rotations of $\mathbb{R}^n$.

Exercise 2.41. If we relax the restriction $\det A = 1$ in the above example we get the **orthogonal groups** O_n. Show that $O_n(\mathbb{R})$ is disconnected.

Exercise 2.42. Show that $\mathrm{SO}_2(\mathbb{R})$ is topologically a circle, and $\mathrm{SO}_2(\mathbb{C})$ is the torus $\mathbb{C}^*$.

Example 2.43. The **symplectic groups** $\mathrm{Sp}_{2n}(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ are the stabilizers of any fixed *symplectic form* on $\mathbb{R}^{2n}$ or $\mathbb{C}^{2n}$ respectively. A nondegenerate bilinear form $\langle , \rangle$ is *symplectic* if it satisfies *skew-symmetry*:

$$\langle v, w\rangle = -\langle w, v\rangle$$

for all w, v.

Exercise 2.44. Show that the skew-symmetry property is equivalent to the condition that $\langle v, v\rangle = 0$ for all v.

Exercise 2.45. Show that if $\mathbb{R}^m$ or $\mathbb{C}^m$ has a symplectic form then m must be even. (Hint: the nondegenerate condition must be used.)

The canonical example of a symplectic form is $\langle v, w\rangle = v^T \Omega w$ where

$$\Omega = \begin{pmatrix} &&&&&&&1\\ &&&&&&1&\\ &&&&&1&&\\ &&&&1&&&\\ &&&-1&&&&\\ &&-1&&&&&\\ &-1&&&&&&\\ -1&&&&&&& \end{pmatrix}$$

(here the example is just shown for $n = 4$, but in general it is a block matrix where the lower left block is a reverse diagonal matrix of all -1's and the upper right is reverse diagonal with 1's, and the rest 0's).

Exercise 2.46. Show that the condition on a matrix M being in Sp_{2n} is that $M^T \Omega M = \Omega$.

Remark 2.47. The symplectic group arises naturally in quantum mechanics. Suppose a system of n quantum particles have positions $x_1, \ldots, x_n$ and momentum values $p_1, \ldots, p_n$ respectively. Think of these values as defining a point $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ in phase space. Define the operator q_i to be the operator that returns the value of the i-th coordinate in this phase space, so it would return x_i if $i \le n$ and p_{i-n} if $i > n$. Then the commutators of these operators satisfy (up to appropriate scalars)

$$q_i q_j - q_j q_i = [q_i, q_j] = \Omega_{ij}$$

where Ω is the matrix defined above. This is a mathematical encoding of *Heisenberg's uncertainty principle*, which states that a particle's position and momentum can never be simultaneously known.

Example 2.48. The **unitary group** $\mathrm{U}_n(\mathbb{C})$ is the group of matrices $A \in \mathrm{GL}_n(\mathbb{C})$ satisfying $A^* = A^{-1}$ where A^* is the *conjugate transpose* obtained by conjugating the coordinates and transposing the matrix.

Note that the equations defining it are not polynomial in the complex coordinates, but they are in the underlying real coordinates. It turns out that U_n (and its special part ${}_n(\mathbb{C})$ of determinant 1) are real Lie groups but not complex Lie groups.

Example 2.49. An example of a non-matrix Lie group is an **elliptic curve** along with its group law. An example of an elliptic curve is the curve given by $y^2 = x^3 - x + 1$ plus the point at infinity given by $(0 : 1 : 0)$ in projective coordinates (obtained by the homogenized equation $zy^2 = x^3 - z^2x + z^3$ in the projective plane).

Given two non-infinity points P and Q on the curve, the line joining them intersects the curve at a unique third point R (which may be infinity if the line is vertical). In this case we define $P + Q = R$. If O is the point at infinity, we define it to be the identity element by setting $O + P = P$ for all P. This gives an abelian Lie group structure on the projective curve, and it is not a matrix Lie group because all matrix Lie groups are *affine varieties*, and elliptic curves are not.

2.2.3 Representations of Lie groups

Definition 2.50. A (finite-dimensional) **representation** of a real or complex Lie group G is a map of Lie groups $G \to \mathrm{GL}(V)$ where V is a finite-dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$ respectively. That is, the map is differentiable on the underlying manifold structures, and is a group homomorphism.

As in the case of representations of finite groups, we may also refer to the representation as a vector space V rather than the map $G \to \mathrm{GL}(V)$, where the action on V is implied.

Definition 2.51. The **character** of a representation $\rho : G \to GL(V)$ of a Lie group is the map $\chi_\rho : T \to \mathbb{C}$ defined by $\chi_\rho(t) = \mathrm{tr}(t)$. Here T is the maximal torus of G (for instance, diagonal matrices in the case of GL_n).

Characters of representations of Lie groups satisfy the following remarkable properties:

1. A representation of a Lie group is uniquely determined by its character.

2. Characters are additive with respect to direct sum: $\chi_V + \chi_W = \chi_{V\oplus W}$. Thus, writing a character in terms of characters of irreducible representations corresponds to decomposing a representation into irreducibles.

3. Characters are multiplicative with respect to tensor product: $\chi_V \chi_W = \chi_{V\otimes W}$

As we have seen, many useful maps between matrix groups can be described by polynomial equations in the variables. These give particularly nice representations of Lie groups.

Definition 2.52. A **polynomial representation** of a matrix Lie group G is a map $G \to \mathrm{GL}(\mathbb{C}^n)$ (or to $\mathrm{GL}(\mathbb{R}^n)$) in which the matrix entries in the image are given by polynomials in the entries of G.

Example 2.53. As we shall see later, there is one irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ for each partition λ having at most n parts. The remaining (non-polynomial) irreducible representations are determined by tensoring with a negative power of the *determinant representation* formed by sending each matrix to the 1×1 matrix consisting of its determinant. Thus understanding the irreducible polynomial representations suffices for understanding the representation theory of $\mathrm{GL}_n(\mathbb{C})$.

The polynomial representation V^λ of $\mathrm{GL}_n(\mathbb{C})$ corresponding to the partition λ has character given by the Schur function $s_\lambda(x_1, \ldots, x_n)$ in n variables, where we think of an element of the torus as a diagonal matrix with $x_1, \ldots, x_n$ on the diagonals.

It follows that all of the combinatorics of Schur functions that we developed last semester is precisely what we need to understand the representation theory of $\mathrm{GL}_n(\mathbb{C})$. We will prove these assertions later in the course.

2.3 Lie algebras and their representations

Roughly speaking, a Lie algebra captures the "local differential information" at the identity element of a Lie group.

To motivate Lie algebras, here are some final facts about Lie groups that make the Lie algebra's importance clear:

1. Any connected Lie group is generated by any open neighborhood of the identity element e. In particular, a map of Lie groups is determined by its restriction to a neighborhood of e.
2. A map of Lie groups $G \to H$ is determined by the induced differential map on tangent spaces $T_e(G) \to T_{e'}(H)$ where e, e' are the identity elements of G and H respectively.
3. As a consequence to the above, a representation $G \to \mathrm{GL}(V)$ of Lie groups is determined by the map of tangent spaces
$$T_e(G) \to T_I(\mathrm{GL}(V)).$$

Because of the last fact, in order to understand representations of Lie groups, understanding their tangent spaces at the identity (especially for GL_n) is all we need. This reduces it to a linear problem which is much easier to understand.

2.3.1 The epsilon method

How does one compute the tangent space to a Lie group at the identity? We simply enforce that the elements "very close" to the identity in the tangent space are "in" the Lie group, as follows.

Definition 2.54. Define the indeterminant ϵ by the relation $\epsilon^2 = 0$, similar to how we can define the imaginary number i to satisfy $i^2 = -1$. In other words, we will work over the ring of coefficients $\mathbb{C}[\epsilon]/(\epsilon^2)$.

We now define the tangent space formally for *matrix Lie groups* only.

Definition 2.55. For a matrix Lie group $G \subseteq \mathrm{GL}_n(\mathbb{C})$ defined by polynomial equations $f_1, \ldots, f_m = 0$, the tangent space $T_I(G)$ at the identity matrix I is the set of matrices X such that
$$I + \epsilon X$$
satisfies the equations $f_1, \ldots, f_m$ over the extended coefficient ring $\mathbb{C}[\epsilon]/(\epsilon^2)$.

Example 2.56. The tangent space $T_I(\mathrm{SL}_n(\mathbb{C}))$ is the set $\{X : \det(I+\epsilon X) = 1\}$. An explicit computation using the definition of ϵ shows that $\det(I+\epsilon X) = 1+\epsilon \mathrm{tr} X$, and so the condition is equivalent to $\mathrm{tr} X = 0$. Thus $\mathfrak{sl}_n(\mathbb{C}) := T_I(\mathrm{SL}_n(\mathbb{C})) = \{X : \mathrm{tr} X = 0\}$.

Example 2.57. A similar analysis to the above shows that the Lie algebra corresponding to $\mathrm{GL}_n(\mathbb{C})$, denoted $\mathfrak{gl}_n(\mathbb{C})$, is simply the set of all $n \times n$ matrices (with no restrictions).

Now, notice that $\mathfrak{sl}_n(\mathbb{C})$ is not closed under matrix multiplication. In general Lie algebras do not have a well-defined product. Indeed, we can see that under the epsilon method, multiplication in the Lie group turns into addition in the Lie algebra:

$$(I + \epsilon X)(I + \epsilon Y) = I + \epsilon(X + Y).$$

However, we do have a well-defined **commutator**. Indeed, if we consider the Lie group commutator $ghg^{-1}h^{-1}$, we can derive an analogous operation on the Lie algebra by using the epsilon method on both g and h separately using two independent commuting indeterminants ϵ and σ, both of which square to 0, as follows.

$$\begin{aligned}(I+\epsilon X)(I+\sigma Y)(I+\epsilon X)^{-1}(I+\epsilon Y)^{-1} &= (I+\epsilon X)(I+\sigma Y)(I-\epsilon X)(I-\epsilon Y)\\ &= (I+\epsilon X+\sigma Y+\epsilon\sigma XY)(I-\epsilon X-\sigma Y+\epsilon\sigma XY)\\ &= I-\epsilon\sigma XY-\epsilon\sigma YX+\epsilon\sigma XY+\epsilon\sigma XY\\ &= I+\epsilon\sigma(XY-YX)\\ &= I+\epsilon\sigma[X,Y]\end{aligned}$$

where $[X, Y] = XY - YX$.

Exercise 2.58. Show that $[X, Y] = XY - YX$ is a well-defined bracket on $\mathfrak{sl}_n(\mathbb{C})$, that is, that $\operatorname{tr}(XY - YX) = 0$ for any matrices $X, Y \in \mathfrak{sl}_n(\mathbb{C})$.

We now have finally motivated the abstract definition of a Lie algebra.

Definition 2.59. A **Lie algebra** is a vector space $\mathfrak{g}$ along with a bilinear **Lie bracket** $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

1. **Skew-symmetry**: $[X, Y] = [Y, X]$
2. **Jacobi identity**: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

It is not hard to verify that the commutator $[X, Y] = XY - YX$ satisfies the above two identities.

Here is the main theorem on the relationship between Lie groups and Lie algebras that completes our story.

Theorem 2.60. *A vector space $\mathfrak{g}$ with a Lie bracket $[,]$ is a tangent space $T_e(G)$ of some Lie group G if and only if it is a Lie algebra. Moreover, the connected component of e of G is uniquely determined by $\mathfrak{g}$.*

In other words, **there is a one-to-one correspondence between Lie algebras and connected Lie groups**.

For the finite dimensional setting, a remarkable theorem shows that in fact we can always assume that the Lie bracket $[,]$ is ordinary commutator of matrices.

Theorem 2.61. *(Ado's Theorem). Every finite-dimensional Lie algebra is isomorphic to a matrix Lie algebra with the commutator bracket.*

A **map** of Lie algebras is a map $f : \mathfrak{g} \to \mathfrak{h}$ that is compatible with their Lie brackets:

$$[fg_1, fg_2] = [g_1, g_2].$$

Using this definition we can define a representation of a Lie algebra.

Definition 2.62. A **representation** of a Lie algebra is a map $\rho : \mathfrak{g} \mapsto \mathfrak{gl}_n(\mathbb{C})$ of Lie algebras.