Math 502: Combinatorics
Homework 3

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] **Drawing fun!** Write out all of the Knuth equivalence classes on the 24 permutations of length 4, along with their insertion tableaux.

2. **The 'e' Pieri rule.** In this problem we will prove the Pieri rule for e's and use it to finish the proof that $\omega s_\lambda = s_{\lambda^T}$. (You may hand in later parts of this problem without doing the ones that precede it, and simply use the statement of the previous parts as fact.)

   - (2) [3 points] Suppose $b > a$ and one computes $T \leftarrow b \leftarrow a$, in other words, inserting b into T with the RSK insertion algorithm and then inserting a into the result. Show that the insertion path of a lies weakly left of the insertion path of b.
   - (1) [1 point] Use the previous part to show that the final (top) square of the insertion path of a is strictly above and weakly left of that of b.
   - (1+) [2 points] Explain why inserting the entries of a column-shaped SSYT C from top to bottom into a general SSYT T results in a tableau S whose shape extends T by a vertical strip.
   - (1+) [2 points] Deduce the "e" Pieri rule from the previous part, which states that

     $$e_r \cdot s_\lambda = \sum_{\nu/\lambda \in \mathrm{Vert}(r)} s_\nu$$

     where $\mathrm{Vert}(r)$ is the set of all vertical strips of size r. (Hint: You may want to follow the proof of the "h" Pieri rule given in the lecture notes.)
   - (1+) [2 points] Prove that

     $$e_\mu = \sum_\lambda K_{\lambda^T \mu} s_\lambda$$

     where $K_{\lambda^T \mu}$ is the Kostka number counting the number of semistandard Young tableaux of shape λ^T and content μ. Do this by induction on the number of parts of μ and using the "e" Pieri rule.

3. (1+) [2 points] **Forgotten expansions:** Let $\{f_\lambda\}$ be the forgotten basis, defined as $f_\lambda = \omega m_\lambda$. Prove that $s_\lambda = \sum_\mu K_{\lambda^T \mu} f_\mu$, and explain how this relates to the identity in the last part of the previous problem via the Hall inner product.

4. (2) [3 points] **Increasing and decreasing subsequences:** Let π be a permutation in S_n, and let ℓ be the length of the longest increasing subsequence of π and d the length of the longest decreasing subsequence of π. Show that $\ell \cdot d \geq n$.

5. (2+) [4 points] **Tableaux major index:** Consider the maj statistic on permutations, defined by

   $$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d.$$

   In other words, it is the sum of the *descents* - the indices d such that the dth entry of the permutation is greater than the $(d+1)$st.

   Define the maj statistic on standard Young tableaux by defining a descent of T to be an entry d of T such that $d+1$ occurs in a lower row than d in T.

   Prove that, if $\mathrm{RSK}(\pi) = (S, T)$, where S is the insertion tableau and T is the recording tableau, then $\mathrm{maj}(\pi) = \mathrm{maj}(T)$.

6. (3-) [8 points] **Knuth equivalence on a necklace:** Consider all arrangements of the numbers $1, 2, \ldots, n$ around a necklace (up to rotation). A *cyclic Knuth move* consists of taking three consecutive letters a, b, c around the necklace in either the clockwise or counterclockwise direction, and if $a < c < b$ or $b < c < a$, switching a and b. Two arrangements are *cyclically Knuth equivalent* if one can be obtained from the other by a sequence of cyclic Knuth moves.

   Show that all arrangements are equivalent to each other with respect to cyclic Knuth equivalence.