Math 502: Combinatorics
Homework 10

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1+) [2 points] How many matroids are there on the set $E = \{1, 2, 3\}$ up to isomorphism (relabeling)?

2. (1+) [2 points] Prove that the *uniform matroid* $U_{m,n} = ([n], \binom{[n]}{m}])$ is a matroid according to the basis definition.

3. (2-) [3 points] Let G be a graph with edge set E, and let $\mathcal{C}$ be the set of all cycles of G, considered as subsets of E. Prove that $(E, \mathcal{C})$ satisfies the circuit axioms C1–C3 defining a matroid.

4. (2+) [4 points] Prove that the Fano matroid, defined as the matroid on $\{1, 2, \ldots, 7\}$ whose bases are all 3-element sets that are **not** collinear in the Fano plane, is not a graphical matroid.

5. (2) [3 points] Prove that if $M = (E, \mathcal{I})$ is a matroid with respect to the independence axioms, and $\mathcal{C}$ is the set of circuits of M, then $(E, \mathcal{C})$ satisfies the circuit axioms (C1)-(C3).

6. (2+) [4 points] Prove that if $M = (E, \mathcal{C})$ is a matroid with respect to the circuit axioms, and $\mathcal{I}$ is the set of subsets of E that contain no member of $\mathcal{C}$, then $(E, \mathcal{I}$ satisfies the independence axioms (I1)-(I3).

   (Hint: You may want to use proof by contradiction as we did in class.)

7. (3-) [8 points] Let A be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

   and let $M_2(A)$ be the representable matroid whose vectors are the column vectors considered as vectors in $\mathbb{F}_2^3$, where $\mathbb{F}_q$ is the finite field of order q. (Note that a matroid described by the column vectors of a matrix has bases given precisely by the invertible maximal minors.) Let $M_3(A)$ be the representable matroid whose vectors are the column vectors of A considered in $\mathbb{F}_3^2$. Show that:

   - The sets of circuits of $M_2(A)$ and $M_3(A)$ are different.
   - $M_2(A)$ is graphic but $M_3(A)$ is not.
   - $M_2(A)$ is representable over $\mathbb{F}_3$, but $M_3(A)$ is not representable over $\mathbb{F}_2$