Skew Schur functions

Def: The skew Schur function $s_{\lambda/\mu}$ is

$$s_{\lambda/\mu} = \sum_{T \in SSYT(\lambda/\mu)} x^T .$$

(shape)

Ex: $s_{(3,2)/(1)}$

$(3,2)/(1)$

$$s_{(3,2)/(1)} = \underset{\begin{smallmatrix} & 1 & 2 \\ 1 & 1 & \end{smallmatrix}}{x_1^3 x_2} + \underset{\begin{smallmatrix} & 2 & 2 \\ 1 & 2 & \end{smallmatrix}}{x_1 x_2^3} + \cdots + \underbrace{2x_1^2x_2^2}_{\begin{smallmatrix} & 1 & 2 \\ 1 & 2 & \end{smallmatrix} \quad \begin{smallmatrix} & 2 & 2 \\ 1 & 1 & \end{smallmatrix}} + \cdots$$

$$+ \underbrace{3m_{(2,1,1)}}_{\begin{smallmatrix} & 2 & 3 \\ 1 & 1 & \end{smallmatrix} \quad \begin{smallmatrix} & 1 & 2 \\ 1 & 3 & \end{smallmatrix} \quad \begin{smallmatrix} & 1 & 3 \\ 1 & 2 & \end{smallmatrix}} + \underbrace{5m_{(1,1,1,1)}}_{\begin{smallmatrix} & 3 & 4 \\ 1 & 2 & \end{smallmatrix} \quad \begin{smallmatrix} & 2 & 4 \\ 1 & 3 & \end{smallmatrix} \quad \begin{smallmatrix} & 2 & 3 \\ 1 & 4 & \end{smallmatrix} \quad \begin{smallmatrix} & 1 & 3 \\ 2 & 4 & \end{smallmatrix} \quad \begin{smallmatrix} & 1 & 4 \\ 2 & 3 & \end{smallmatrix}}$$

$$= m_{(3,1)} + 2m_{(2,2)} + 3m_{(2,1,1)} + 5m_{(1,1,1,1)}$$

$$s_{(2,2)} = m_{(2,2)} + m_{(2,1,1)} + 2m_{(1,1,1,1)}$$

$$s_{(3,1)} = m_{(3,1)} + m_{(2,2)} + 2m_{(2,1,1)} + 3m_{(1,1,1,1)}$$

Hwk:
- Chrom. symm.
- Macdonald positivity
- $s(1,q,q^2 \ldots)$

$$\Rightarrow \quad s_{(3,2)/(1)} = s_{(2,2)} + s_{(3,1)} .$$

Thm: Skew schur functions are Schur positive. Moreover, $s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_\nu$, where $c^{\lambda}_{\mu\nu}$ are the Littlewood-Richardson coefficients w/ a combinatorial interpretation known as the Littlewood-Richardson rule.

Def: A word w is <u>ballot</u> or <u>Yamanouchi</u> or <u>lattice</u>

if every suffix $w_i, w_{i+1}, \ldots, w_n$ has

$$\left.\begin{array}{l} \#1\text{'s} \geq \#2\text{'s} \\ \#2\text{'s} \geq \#3\text{'s} \\ \vdots \end{array}\right\} \text{ i.e. partition}$$

Def: A skew tableau is <u>Littlewood-Richardson</u> if its reading word is ballot.

Thm: $c^{\lambda}_{\mu\nu} = \#$LR tableaux with shape λ/μ and content ν. $\boxed{\text{(Knutson-Tao, etc for possible project)}}$

Proofs via <u>crystal</u> <u>base</u> <u>theory</u> ($\leftrightarrow$ reps of $U_q(sl_n)$ at $q \to 0$)

<u>Crystals on words</u>

Def: Let w be a word of 1's and 2's.

The <u>raising</u> <u>operator</u> E_1 and <u>lowering</u> <u>operator</u> F_1 are defined on such words by:

- Replace all 1's w/ closed parentheses)
- Replace all 2's w/ open parentheses (
- Pair off parentheses as much as possible
- Unpaired parentheses look like

$$\begin{array}{c}))))\cdots)((((\cdots(\\ 11\cdots1\,222\cdots2 \end{array}$$

$E_1(w)$ formed by changing first unmatched 2 to a 1

$F_1(w)$ formed by changing last unmatched 1 to 2

If we can't apply them, $E_1(w) = \emptyset$, or $F_1(w) = \emptyset$

$\emptyset$

$\uparrow$

Ex:

1

1 1 2 2 1 2 1 1 1 1 1 1 2 1 1 1 1 ← top of chain: Ballot! (hwk)

$\downarrow\uparrow$

1 1 2 2 1 2 1 1 1 1 1 1 2 1 1 1 2

$\downarrow\uparrow$

1 1 2 2 1 2 1 1 1 1 1 1 2 1 1 2 2

$\downarrow\uparrow$

1 1 2 2 1 2 1 1 1 1 1 1 2 1 2 2 2

$\downarrow\uparrow$

1 1 2 2 1 2 1 1 1 1 1 2 2 1 2 2 2

$F_1 \downarrow \uparrow F_1$

1 1 2 2 1 2 1 1 1 1 2 2 2 1 2 2 2 ← START

)) (( ) ( )))) ((( ) (((

$\downarrow\uparrow$

1 1 2 2 1 2 1 1 1 2 2 2 2 1 2 2 2

$\downarrow\uparrow$

1 1 2 2 1 2 1 1 2 2 2 2 2 1 2 2 2

$\downarrow\uparrow$

1 2 2 2 1 2 1 1 2 2 2 2 2 1 2 2 2

$\downarrow\uparrow$

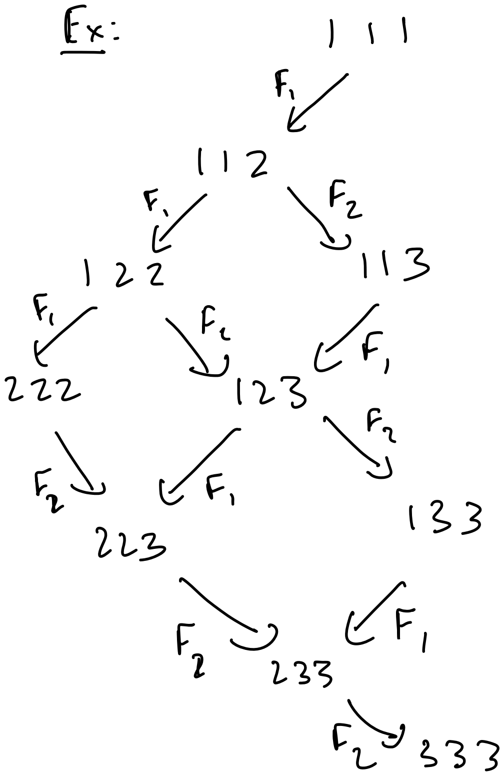
2 2 2 2 1 2 1 1 2 2 2 2 2 1 2 2 2 ← "antiballot" (hwk)

$\downarrow$

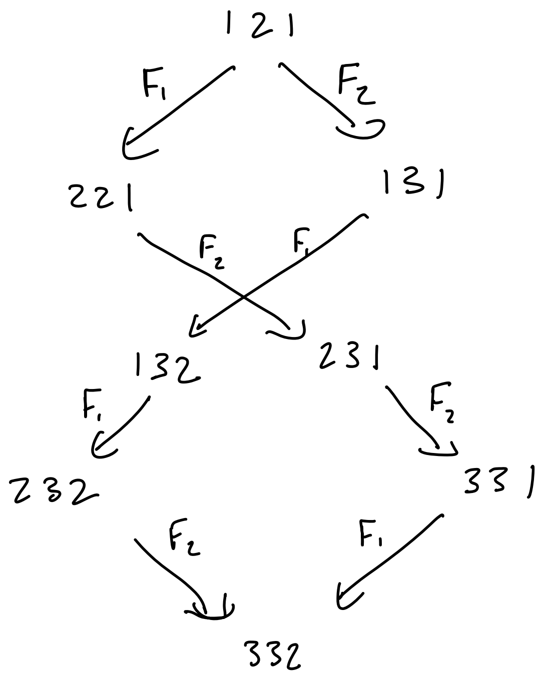
$\emptyset$

Def: If w any word, $E_1(w)$ and $F_1(w)$ are formed by applying E_1, F_1 to 1/2 subword.
E_i, F_i : same for i's and $(i+1)$'s.

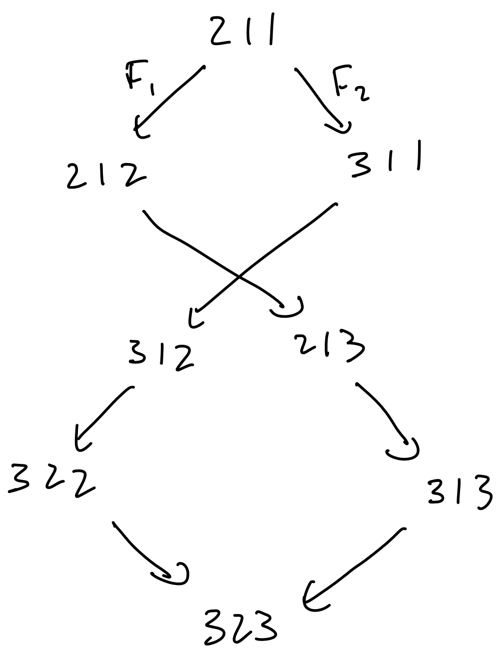
Ex:



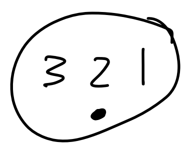
111
112
122
113
222
123
223
133
233
333
F_1
F_2



121
221
131
132
231
232
331
332
F_1
F_2



211
212
311
312
213
322
313
323
F_1
F_2



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Def: A word is <u>highest weight</u> if $E_i(w)=0$ for all i.

Lemma: w is highest weight iff it is ballot.

Pf: (Hwk).

Thm: E_i, F_i are well-defined on skew SSYT's (by acting on the reading word) and are compatible with JDT slides and Knuth equivalence.

Pf: E_i and F_i only affect parts of the $i, i+1$ subtableau that are one row high:

```
   1 1 2 2 2
        1 ~~3~~ 1 1 1 2 2
                      1 2
```

← Columns get paired in parenthesization.

So the result is still semistandard (since) all other letters are still $> i+1$ or $< i$.)

<u>Compatibility with JDT</u>: Suffices to check E_1, F_1 compatibility on a slide through the 1,2-subtableau, since i case is similar and other letters are unchanged.

Ex:

$$\begin{array}{ccc}
\begin{array}{l} \;\;1\,1\,2\,2 \\ \times\,1\,1\,2 \end{array} & \xrightarrow{\text{JDT}} & \begin{array}{l} 1\,2\,2 \\ \;\;1\,1\,1\,2 \end{array} \\
\Big\downarrow F_1 & & \Big\downarrow \\
\begin{array}{l} \;\;1\,2\,2\,2 \\ \times\,1\,1\,2 \end{array} & \xrightarrow{\text{JDT}} & \begin{array}{l} 1\,2\,2 \\ \;\;1\,1\,2\,2 \end{array}
\end{array}$$

Can check that the above is the only interesting case for F_1: when the element above the x is the 1 that is changed.

<u>Compatibility with Knuth moves</u>: Suppose $F_i(w)$ changes an i to an $i+1$. Then it certainly commutes with all Knuth moves not involving the changed letter. So consider a Knuth move that could involve the changed letter:

<u>Case 1</u>: $i=a$, b on right (cf $a<b<c$ in Knuth move)

$$\begin{tikzcd}
\cdots i\,c\,b \cdots \arrow[r, leftrightarrow, "\text{Knuth move}"] \arrow[d, "F_i"] & c\,i\,b \arrow[d, "F_i"] \\
i{+}1\;c\;b \arrow[r, leftrightarrow] & c\;i{+}1\;b
\end{tikzcd}$$

with $b \geq i$, $c>b$

If $b \geq i+1$ this commutes.

Otherwise if $b=i$, then if $c>i+1$ then b is the letter F_i changes instead of the first i, $\rightarrow\leftarrow$. So if $b=i$ then $c=i+1$ and we have the diagram

$$\begin{tikzcd}
i\;i{+}1\;i \arrow[r, leftrightarrow] \arrow[d, "F_i"] & i{+}1\;i\;i \arrow[d, "F_i"] \\
i{+}1\;i{+}1\;i \arrow[r, leftrightarrow] & i{+}1\;i\;i{+}1
\end{tikzcd}$$

which also commutes, using the Knuth move thinking of the <u>left</u> element as b in the bottom row.

<u>Case 2</u>: $i=c$, b on right

$$\begin{tikzcd}
i\,a\,b \arrow[r, leftrightarrow] \arrow[d, "F_i"] & a\,i\,b \arrow[d, "F_i"] \\
i{+}1\;a\;b \arrow[r, leftrightarrow] & a\;i{+}1\;b
\end{tikzcd}$$

with $i>b\geq a$

This commutes since $i+1>b$ too.

Case 3: $i = a$, b on left:

$$\begin{tikzcd} b\ i\ c \arrow[r, leftrightarrow] \arrow[d, "F_i"] & b\ c\ i \arrow[d, "F_i"] \\ b\ i{+}1\ c \arrow[r, leftrightarrow] & b\ c\ i{+}1 \end{tikzcd}$$

Note $b \neq i+1$ by F_i changing the i, so $b > i+1$, $c > b$
$\Rightarrow$ commutes

Case 4: $i = c$, b on left:

$$\begin{tikzcd} b\ a\ i \arrow[r, leftrightarrow] \arrow[d, "F_i"] & b\ i\ a \arrow[d, "F_i"] \\ b\ a\ i{+}1 \arrow[r, leftrightarrow] & b\ i{+}1\ a \end{tikzcd}$$

$a < b \leq i$

so $a < b < i+1$ ✓

Case 5: $i = b$ on left

$$\begin{tikzcd} i\ a\ c \arrow[r, leftrightarrow] \arrow[d, "F_i"] & i\ c\ a \arrow[d, "F_i"] \\ i{+}1\ a\ c \arrow[r, leftrightarrow] & i{+}1\ c\ a \end{tikzcd}$$

commutes if $c \geq i+1$

If $c = i$ instead, F_i changes c, not i, $\Rightarrow\Leftarrow$.

Case 6: $i = b$ on right:

$$\begin{tikzcd} a\ c\ i \arrow[r, leftrightarrow] \arrow[d, "F_i"] & c\ a\ i \arrow[d, "F_i"] \\ a\ c\ i{+}1 \arrow[r, leftrightarrow] & c\ a\ i{+}1 \end{tikzcd}$$

$a \leq i < c$

$c \neq i+1$ by F_i application, so $c \geq i+2$, so it commutes.

$\square$

Lemma: For a fixed shape λ, the SSYT's of shape λ all lie in a single connected component of the crystal graph w/ E_i, F_i as edges, and there is a unique highest element,

$$T_\lambda \quad \begin{array}{|c|c|c|c|c|} \hline 4 \\ \cline{1-1} 3 & 3 & 3 & 3 \\ \cline{1-4} 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \; ,$$

on which all E_i's evaluate to $\emptyset$.

Pf: We show that if an SSYT T is not the T_λ tableau drawn above, then we can apply some E_i; this will suffice.

Consider the lowest row, say row r, that contains an element greater than r (which must exist since $T \neq T_\lambda$).

$$\begin{array}{lllll} 5 \\ 3 & 3 & 4 & 6 \\ 2 & 2 & 3 & 4 & \leftarrow r \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

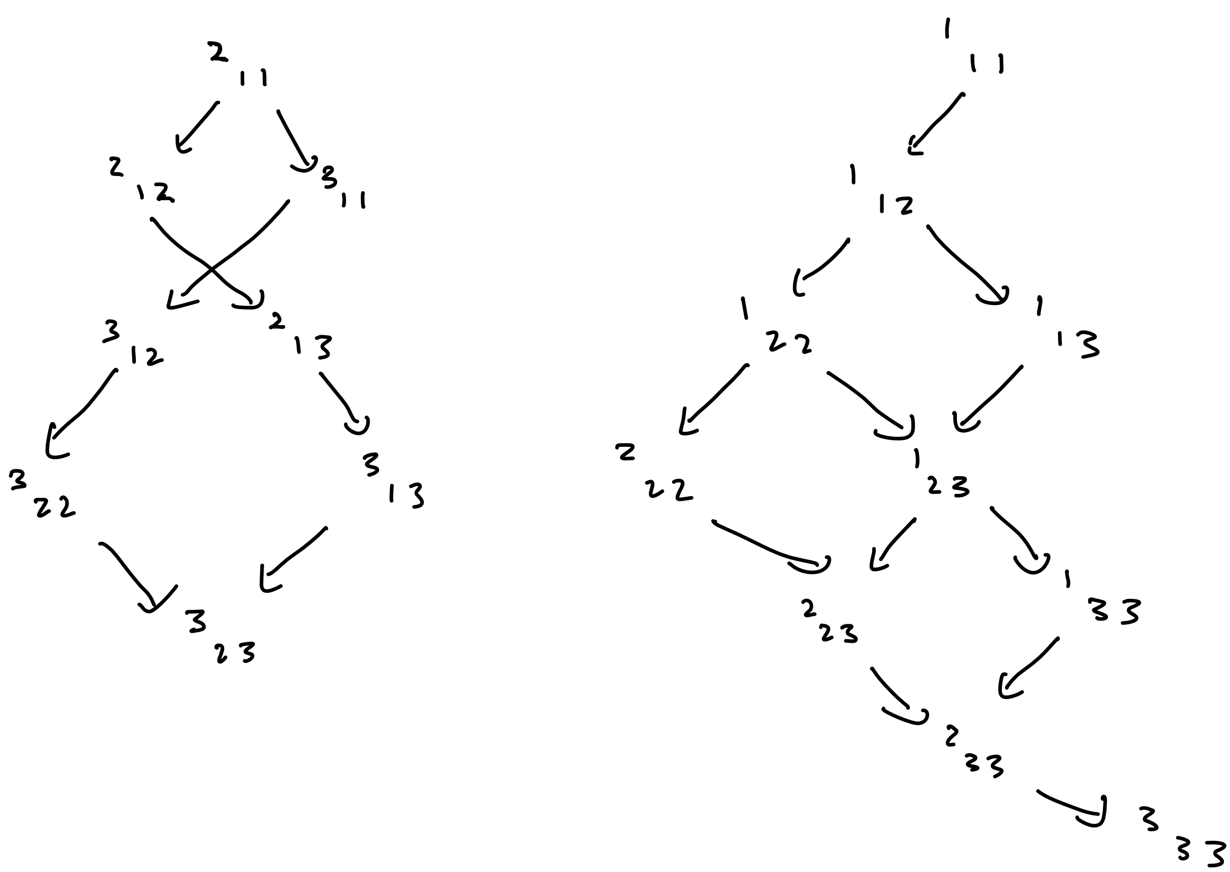
Let m be the first letter in row r s.t. $m > r$. Then there is no $m-1$ after this m in the reading word, by semistandardness in row r and the fact that all letters in lower rows are $\leq r-1$.

Thus we can apply E_{m-1} to the $(m, m-1)$ subword

$\square$

Cor: Every connected component of a crystal of all skew tableaux of a given shape λ/μ is isomorphic as a directed graph to a straight shape crystal, formed by JDT rectifying all of the elements.

Ex: $s_{(3,1)/(1)}$ crystal:



$= s_{(2,1)} + s_{(3)}$

Cor: # times Schur function crystal s_ν appears in $s_{\lambda/\mu}$ is # highest weight elts that rectify to T_ν = # SSYT's shape λ/μ that rectify to ν.

= # for which E_i's are all $\emptyset$

= # ballot SSYT's of shape λ/μ with content ν.

Littlewood-Richardson rule follows. QED.