Lecture 3 $[n] := \{1,2,3,\dots,n\}$

Combinatorial interpretations/definitions

(Defining numerical expressions as cardinalities)

Symbol	Combinatorial def
n	Size of an n-elt set, like $\lvert [n] \rvert$
$a-b$	Remove b things from a things $\lvert A \setminus B \rvert$ where $B \subseteq A$, $\lvert A\rvert = a$, $\lvert B \rvert = b$
$a+b$	Add b things to a things (a or b) $\lvert A \sqcup B \rvert$ (CASEWORK)
$a \cdot b$	One of a choices, then one of b choices; $\lvert A \times B \rvert$
a/b	Sort a things into groups of size b; how many groups? $A = B_1 \sqcup B_2 \sqcup \dots \sqcup B_{a/b}$ $\lvert B_i \rvert = \lvert B_j \rvert \ \forall i,j$
$n!$	Rearrangements of n things in order $\lvert S_n \rvert = \lvert \{\pi : [n] \to [n] \text{ bijection}\} \rvert$
2^n	Binary sequences of length n; $\lvert \{0,1\}^n \rvert$ Subsets of $[n]$ $\lvert \mathcal{P}([n]) \rvert$

Q: Find a bijection $\{0,1\}^n \xrightarrow{\cong} \mathcal{P}([n])$.

Q: Give a combinatorial proof that

comb def of mult

$$2^n = \underbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}_{n}$$

above definition

Def: A combinatorial proof of an equality $a=b$ is:

- Finding a set A s.t. $|A| = a$
- Finding a set B s.t. $|B| = b$
- Establishing a bijection $A \to B$.

If $A = B$, we say it is a proof by counting in two ways.

Ex: Combinatorial proof that $2^a \cdot 2^b = 2^{a+b}$

The left hand side (LHS) counts the pairs (w_1, w_2) of binary strings of length a, b respectively.

The right hand side (RHS) counts the binary strings of length $a+b$.

The bijection $(w_1, w_2) \mapsto w_1 w_2$ given by concatenation shows these are equivalent. $\square$

Combinations

Symbol	Comb. def
$\binom{n}{k}$, "n choose k"	# ways to choose k things from n distinct things in no particular order; $\binom{[n]}{k} = \{X \subseteq [n] : \lvert X \rvert = k\}$
$\left(\!\binom{n}{k}\!\right)$, "n choose k with repeats"	# ways to choose k things from n things where you can choose the same thing more than once (vending machine problem; multisets) $\left(\!\binom{[n]}{k}\!\right) = \{\underline{\text{multisets}} \text{ of \#s from } [n] \text{ of size } k\}$ Multiset: $\{1, 1, 2, 3, 3, 5, 7, 7, 7\}$ Rigorously: $\{1_0, 1_1, 2_0, 3_0, 3_1, 5_0, 7_0, 7_1, 7_2\}$
n^k	# ways to choose k things from n <u>in order</u>, repeats allowed (sequences)
$(n)_k$	# ways to choose k things from n in order, repeats not allowed (sequences of distinct elts)

Combining combinatorial defs:

<u>Ex:</u> $\binom{\binom{n}{k}}{m}$: # ways to choose m k-elt subsets of $[n]$.

<u>Ex:</u> $2^{\binom{n}{k}}$: # ways to choose some k-elt subsets of $[n]$.

<u>Ex:</u> $n \cdot 2^{n-1}$: # ways to choose one special element from $[n]$ <u>and then</u> choose a subset of the remaining elements.

Combinatorial proofs of basic identities:

In class, we will show:

- $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$
- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, $\binom{n}{0} = \binom{n}{n} = 1$

  (Pascal Recursion)

- $(n)_k \cdot (n-k)! = n!$

  which implies $(n)_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$ "falling factorial"

- $n^k = \underbrace{n \cdot n \cdot n \cdot \cdots \cdot n}_{k}$

- $\binom{n}{k} \cdot k! = (n)_k$

  and therefore $\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!}}$

- $\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$ using M&M's