Schur functions

Motivation: $\{s_\lambda\}$ basis of Λ_R for $R \supseteq \mathbb{Z}$

- Correspond to irred. reps of S_n and GL_n
- Govern geometry of linear intersections

(both: next semester!)

Two defs this week: combinatorial and algebraic.

① Combinatorial def:

Def: A semistandard Young tableau of shape λ is a way of filling the boxes of the Young diagram of λ w/ pos ints that are

- Weakly increasing across rows
- Strictly increasing up columns

Ex: $SSYT((3,3,2))$ contains:

$$\begin{matrix} 4 & & \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{matrix}, \quad \begin{matrix} 3 & & \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{matrix} \quad \cdots$$

$SSYT(\lambda) = \{ SSYT\text{'s of shape } \lambda \}$

For an SSYT, its monomial weight is

$$x_1^{\#1\text{'s}} x_2^{\#2\text{'s}} \cdots = x^T$$

So if $T = \begin{array}{llllll} 7 \\ 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 5 \\ 1 & 1 & 1 & 2 & 2 & 3 \end{array}$

$$x^T = x_1^3 x_2^4 x_3^3 x_4^3 x_5 x_7 .$$

<u>Def:</u> <u>Schur</u> <u>functions</u>

$$\boxed{s_\lambda = \sum_{T \in SSYT(\lambda)} x^T .}$$

Finitely many vars $x_1, \ldots, x_n$: limits us to using $1, \ldots, n$ in boxes.

<u>Ex:</u> $s_{\tiny\yng(2,1)} = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$

in 3 vars $\quad \begin{array}{l} 2 \\ 1\,1 \end{array} \quad \begin{array}{l} 2 \\ 1\,2 \end{array} \quad \begin{array}{l} 3 \\ 1\,2 \end{array} \quad \begin{array}{l} 2 \\ 1\,3 \end{array} \quad \begin{array}{l} 3 \\ 1\,1 \end{array} \quad \begin{array}{l} 3 \\ 1\,3 \end{array} \quad \begin{array}{l} 3 \\ 2\,2 \end{array} \quad \begin{array}{l} 3 \\ 2\,3 \end{array}$

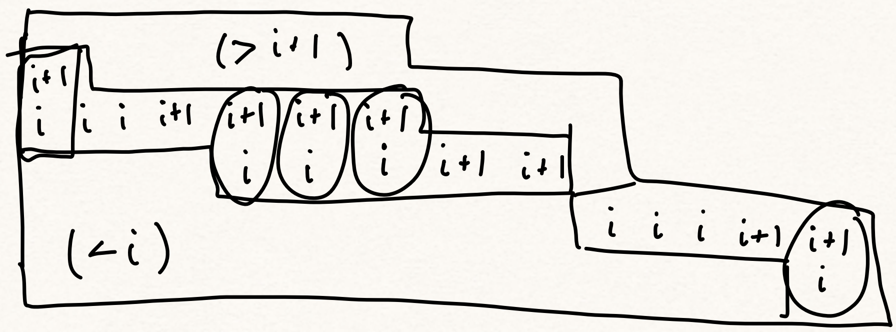
$$= m_{2,1} + 2m_{111} .$$

<u>Why symmetric?</u>

<u>Thm</u>: s_λ is a symmetric function.

<u>Pf</u>: Suffices to show invariant under $x_i \leftrightarrow x_{i+1}$.

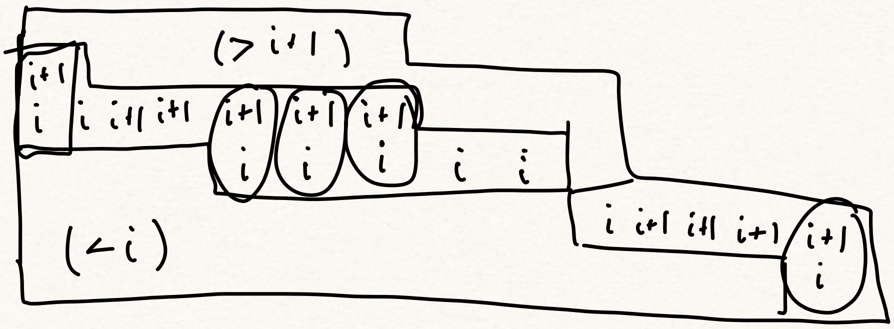
Consider the following involution on Young tableaux. Given SSYT T, consider its $i, i+1$ entries:

$T =$ 

Notice that in each column, we either have $\boxed{\begin{matrix} i+1 \\ i \end{matrix}}$, just i, just $i+1$, or neither.

And in a row, $\boxed{i\, i\, i\, i\;\, i{+}1\; i{+}1}$'s appear consecutively.

So, cancel the $\boxed{\begin{matrix} i+1 \\ i \end{matrix}}$ blocks, and all remaining rows look like $(i)^k (i+1)^j$. Swap these out for $i^j (i+1)^k$, and we have altogether swapped the # i's w/ # $i+1$'s. This involution therefore shows symmetry. $\square$

$\varphi(T) =$ 

So s_λ's symmetric. **Basis?**

<u>Compare to m_μ's:</u>

<u>Lemma</u>: $s_\lambda = \sum K_{\lambda\mu} m_\mu$ where $K_{\lambda\mu} = \#$ SSYT's w/ shape λ, <u>content</u> μ.

<u>Content</u> $= (\#1\text{'s}, \#2\text{'s}, \ldots)$

so, μ_i i's.

<u>Pf</u>: $K_{\lambda\mu}$ is coeff of x^μ. $\square$

<u>Ex</u>:

$$\begin{array}{cc} & \mu: \ (1,1,1) \quad (2,1) \quad (3) \\ \begin{array}{r} (1,1,1) \\ \lambda \ (2,1) \\ (3) \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

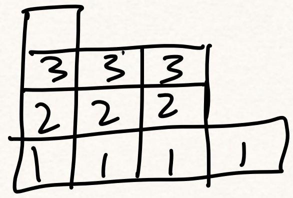
$\leftarrow K_{\lambda\mu}$ matrix for $|\lambda| = |\mu| = 3$

Lemma 1: $K_{\lambda\lambda} = 1$.

Pf: Want to fill λ w/

- λ_1 1's
- λ_2 2's

$\vdots$



The 1's take up entire bottom row
2's " " " next row

and so on. (In general i can only be in rows $1, 2, \ldots, i$). $\square$

Lemma 2: If $K_{\lambda\mu} \neq 0$, then $\lambda \trianglerighteq \mu$.
↑ dominance.

(So, if we extend dominance to a total ordering for λ and μ, K is lower-triangular).

Pf: Suppose $K_{\lambda\mu} \neq 0$. Consider an SSYT of shape λ, content μ:

Ex:
$$\begin{array}{llllll} 3 & 4 & & & & \\ 2 & 2 & 2 & 3 & & \\ 1 & 1 & 1 & 1 & 2 & 4 \end{array}$$

$\lambda = (6, 4, 2)$
$\mu = (4, 4, 2, 2)$

Then $\lambda_1 \geq \mu_1$ since all 1's in bot. row.

$\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$ since all 1's, 2's in first two rows

and so on. So $\lambda \succeq \mu$.

$\square$

<u>Cor</u>: $\{S_\lambda : \lambda \text{ partition}\}$ is a basis for Λ_R for any $R \supseteq \mathbb{Z}$.