Homogeneous and power sum bases

Recall h_λ, p_λ. Want to show they are bases.

Lemma: $h_\lambda = \sum N_{\lambda\mu} m_\mu$ where $N_{\lambda\mu}$ is # of integer matrices w/ row sums λ_i, col sums μ_j.

Pf: Similar to e_λ, $M_{\lambda\mu}$ pf. $\square$

Lemma: $N_{\lambda\mu}$ is always positive if $|\lambda| = |\mu|$.

Pf: Can always find a matrix w/ greedy algorithm: Fill rows from top to bottom, always choosing lex-greatest row that is compatible w/ μ cols. $\square$

Ex:

$$\begin{array}{ccccc|c}
4 & 1 & & & & 5 \\
 & 3 & & & & 3 \\
 & & 1 & 1 & & 2 \\
 & & & & 1 & 1 \\
\hline
4 & 4 & 1 & 1 & 1 &
\end{array}$$

Since $N_{\lambda\mu} > 0$ $\forall \lambda, \mu$, can't use upper-Δ method

Instead: compare h's to e's:

Lem: $e_n = h_1 e_{n-1} - h_2 e_{n-2} + h_3 e_{n-3} - \cdots + (-1)^{n-1} h_n$

Pf: Optional hwk outlines a generating fn proof of this, but here let's do monomial counting and inclusion-exclusion.

Can rewrite:

$$h_n - h_{n-1} e_1 + h_{n-2} e_2 - \cdots + (-1)^n e_n = 0$$

How many times does an arb. monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ appear? (for $|\lambda| = n$)

- Once in h_n
- Once in $h_{n-1} e_1$ for each x_i in the monomial, so contribution of $-k$.
- $\binom{k}{2}$ times from $h_{n-2} e_2$

  e.g. $\left(x_1^{\lambda_1} x_2^{\lambda_2 - 1} x_3^{\lambda_3} \cdots x_{k-1}^{\lambda_{k-1}-1} x_k^{\lambda_k}\right)(x_2 x_k)$

  so choose which 2 vars come from an e_2 term.
- Coeff of $-\binom{k}{3}$ from $h_{n-3} e_3$

and so on, so coeff is

$$1 - k + \binom{k}{2} - \binom{k}{3} + \cdots$$

$$= \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots = 0.$$

✓ $\square$

Now, can use this to recursively start from $e_1 = h_1$ and write either e's in terms of h's or vice versa.

$\Rightarrow$ h_λ's a basis too!

Power sums

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}.$$

Not always a basis:

Ex: $x_1 x_2 = \frac{1}{2}(p_1^2 - p_2)$

need rational coeffs!

R has to contain $\mathbb{Q}$, not just $\mathbb{Z}$.

Thm: If $\mathbb{Q} \subseteq R$, $\{p_\lambda\}$ basis!

Two proofs:

① Let $p_\lambda = \sum a_{\lambda\mu} m_\mu$

Claim: $a_{\lambda\mu} = 0$ whenever $\lambda^* < \mu^*$ in lex order, and $a_{\lambda\lambda} \neq 0$.

Pf: If $\lambda^* < \mu^*$, let i be first position s.t. $\lambda_i^* < \mu_i^*$. Then note $\lambda_i^* = \#$ parts of λ that are $\geq i$

$\mu_i^* = \#$ parts of μ that are $\geq i$.

If $\lambda_i^* < \mu_i^*$ then we only have λ_i^* p_d's in our product w/ $d \geq i$, and μ_i^* exponents $\geq i$ in our monomial, so we can't form our monomial by multiplying out terms. ✓

<u>Second claim</u>: $a_{\lambda\lambda} \neq 0$: can always make $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ in at least one way by picking $x_i^{\lambda_i}$ from p_{λ_i} for all i.

QED

Thus we have an invertible transition matrix, over $\mathbb{Q}$ b/c we don't know all $a_{\lambda\lambda}$'s $= 1$.

$$\begin{array}{c|ccc} a_{\lambda\mu} & m_3 & m_{21} & m_{111} \\ \hline p_3 & 1 & 0 & 0 \\ p_{21} & 1 & 1 & 0 \\ p_{111} & 1 & 3 & 6 \end{array}$$

Second proof:

Newton-Girard Identities

$$p_n - e_1 p_{n-1} + e_2 p_{n-2} - \cdots + (-1)^{n-1} p_1 e_{n-1} + n \cdot (-1)^n e_n = 0$$

extra n factor on last term!

Pf: Consider polynomial

$$p(z) = (z - x_1)(z - x_2) \cdots (z - x_n)$$
$$= z^n - e_1 z^{n-1} + e_2 z^{n-2} - \cdots + (-1)^n e_n$$

$$0 = p(x_1) = x_1^n - e_1 x_1^{n-1} + e_2 x_1^{n-2} - \cdots + (-1)^n e_n$$
$$+ \quad 0 = p(x_2) = x_2^n - e_1 x_2^{n-1} + e_2 x_2^{n-2} - \cdots + (-1)^n e_n$$
$$\vdots \qquad \qquad \vdots$$
$$+ \quad 0 = p(x_n) = x_n^n - e_1 x_n^{n-1} + e_2 x_n^{n-2} - \cdots + (-1)^n e_n$$

$$0 = p_n - e_1 p_{n-1} + e_2 p_{n-2} - \cdots + (-1)^n \cdot n \cdot e_n$$

QED