More bases for Λ

Recall: $m_{(3,2)} = x_1^3 x_2^2 + x_2^3 x_1^2 + x_1^3 x_3^2 + \cdots$

m_λ's are a basis of all symmetric functions Λ.

Def: Elementary symmetric functions

$$e_d = \sum_{i_1 < i_2 < \cdots < i_d} x_{i_1} x_{i_2} \cdots x_{i_d}$$

$$e_\lambda = e_{\lambda_1} \cdot e_{\lambda_2} \cdots e_{\lambda_k} \qquad \lambda = (\lambda_1, \ldots, \lambda_k)$$

Ex: $e_{(2,1)} = e_2 e_1 = (x_1 x_2 + x_1 x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$

$$= m_{(2,1)} + m_{(1,1,1)}$$

Thm (Fundamental thm of Symmetric Function Theory)

$\{e_\lambda\}$ basis of Λ_R over any ring R containing $\mathbb{Z}$.

Lemma: Let $M_{\lambda\mu}$ = # 0-1 matrices w/ row sums λ_i, col sums μ_j.

Then $e_\lambda = \sum_\mu M_{\lambda\mu} m_\mu$.

Pf: $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$. A term from e_{λ_j} is $x_{i_1} x_{i_2} \cdots x_{i_{\lambda_j}}$ which represents a row of 0's and 1's w/ 1's in cols $i_1, i_2, \ldots, i_{\lambda_j}$. Picking one row for each λ_j, want to obtain $x_1^{\mu_1} x_2^{\mu_2} \cdots$

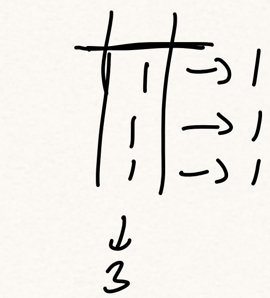
which means we want the col. sums to be μ_j's.

$\square$

So $(M_{\lambda\mu})$ is transition matrix btwn m_λ, e_λ bases for all $\lambda, \mu \vdash d$ for any given d. Want to show it is invertible.

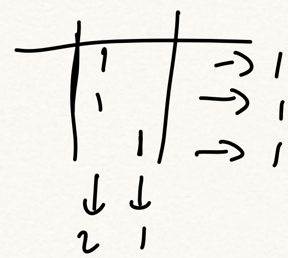
Ex:

	$M_{\lambda\mu}$	μ: (3)	$(2,1)$	$(1,1,1)$
	$(1,1,1)$	1	3	6
λ	$(2,1)$	0	1	3
	(3)	0	0	1



Def: $\lambda \trianglerighteq \mu$ in dominance order

if $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i.



Ex: $(3,2,2,1) \triangleright (3,2,1,1,1)$

Note: lex order refines dominance order to a total ordering.

Lemma: $(M_{\lambda\mu})$ is upper Δ w/ 1's on diagonal when μ's are listed in lex order and λ's are listed in transpose lex order.

Pf: First note $M_{\lambda\lambda^*} = 1$:

$$\begin{array}{cccc|l} 1 & 1 & 1 & 1 & \to 4 \\ 1 & 1 & & & \to 3 \\ 1 & & & & \to 2 \\ \hline \downarrow & \downarrow & \downarrow & \downarrow & \\ 3 & 2 & 1 & 1 & \end{array}$$

Row, col sums transpose means the Young diagram must be filled w/ 1's.

Now we show $M_{\lambda\mu} = 0$ whenever $\lambda^* < \mu$ in lex order. If $\lambda^* < \mu$ in lex order, then λ^* does not dominate μ. Now assume for $\to\leftarrow$ that $M_{\lambda\mu} \neq 0$, so there is a 0-1 matrix:

$$\begin{array}{cccccc|l} 1 & 1 & & 1 & 1 & & \to 4 \\ 1 & & 1 & & & 1 & \to 4 \\ 1 & 1 & 1 & & & & \to 2 \\ \hline \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 3 & 2 & 2 & 1 & 1 & 1 & \end{array} \quad \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \lambda$$

$$\underbrace{3 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1}_{\mu}$$

But λ^* is the col sums formed by shoving all the 1's to the left in each row as much as possible. This dominates μ, a contradiction.

$\square$

<u>Cor:</u> e_λ is a basis (invert $M_{\lambda\mu}$).

Ex: Write $x_1^2+x_2^2+x_3^2+\cdots$ in elementary basis:

$$e_1^2-2e_2 = e_{(1,1)}-2e_{(2)}$$

Ex: Write $x_1^3+x_2^3+x_3^3+\cdots$ in elementary basis:

$$e_{111}-3e_{2,1}+9e_3.$$

(Suffice to use 3 vars for above. Need 4 for deg 4 etc).

Two more bases

- Homogeneous basis:

$$h_d = \sum_{i_1\le i_2\le\cdots\le i_d} x_{i_1}\cdots x_{i_d} = \sum \text{all monoms of deg } d, \text{ squares allowed.}$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2}\cdots h_{\lambda_k}$$

- Power sum basis:

$$p_d = \sum x_i^d$$

$$p_\lambda = p_{\lambda_1}\cdots p_{\lambda_k}.$$

(Show Sage for $m_{2,1}$ in all bases).