Introduction to Symmetric Functions

Q: How many ways to fill in a table with specified row and col sums?

Application to statistics: partial data sets

Ex: Census says, among 20 people in Smalltown:

# commuting by:	# living condition
• Car: 8	• Homeowner: 10
• Bike: 7	• Rental: 7
• Foot: 3	• Other (camper van etc): 3
• Other: 2	

How to figure out how many car commuters are homeowners etc? Bounds? Possibilities?

	car	bike	foot	other	
Home	6	2	2	0	→ 10
Rent	2	5	0	0	→ 7
Other	0	0	1	2	→ 3
	↓ 8	↓ 7	↓ 3	↓ 2	

just one of many ways to fill in the table.

Note: $(10, 7, 3)$ and $(8, 7, 3, 2)$ are partitions of 20.

Need same size for a table to exist.

Symm fns: gen. fns whose index vars are partitions.

Def: A polynomial f in vars $x_1, \ldots, x_n$ is symmetric if $\forall \pi \in S_n$,

$$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

Note: Enough to check swaps of $i, i+1$

Ex: $f(x_1, x_2) = x_1^2 x_2 + x_2^2 x_1$ ← symmetric

$\Rightarrow f(x_2, x_1) = x_2^2 x_1 + x_1^2 x_2$

But $g(x_1, x_2) = x_1^2 + x_2$ ← not symmetric

Ex: $f(x_1, x_2, x_3) = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + 3x_1 x_2 x_3 + 1$

Ex: constants: $0, 1, \sqrt{2}$ are all symmetric

Def: Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

Then monomial symmetric function for λ is

$$m_\lambda(x_1, \ldots, x_n) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} + \cdots \text{similar terms}$$

$$= \left(\sum_{\pi \in S_n} x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(n)}^{\lambda_n} \right) \cdot \frac{1}{\prod m_i!}$$

multiplicities of each part

Ex: f above is $m_{(2,1)} + 3m_{(1,1,1)} + m_0$ in x_1, x_2, x_3.

$m_{(2,1)} = m_{(2,1,0)}$, $\quad m_0 = m_{(0,0,0)}$

Thm: Every symm. poly. f in $x_1, \dots, x_n$ over $\mathbb{Q}$ can be written uniquely as a $\mathbb{Q}$-linear combination of m_λ's for λ partition w/ at most n parts.

Def: $\Lambda_{\mathbb{Q}}(x_1, \dots, x_n) = \{$symm poly's w/ coeffs in $\mathbb{Q}\}$

$\cong \mathbb{Q}[m_\lambda : \lambda$ has at most n parts$]$

Note: $\Lambda_{\mathbb{Q}}(\underline{x_n})$ is an <u>algebra</u> - vector space w/ ring structure

Why? f, g symmetric $\Rightarrow$ $f+g$, $c \cdot f$, fg symmetric.

Lem: $\Lambda_{\mathbb{Q}}(\underline{x_n})$ is a <u>graded</u> algebra:

$$\Lambda_{\mathbb{Q}}(x_1, \dots, x_n) = \bigoplus_{d=0}^{\infty} \underbrace{\Lambda_{\mathbb{Q}}^{(d)}(x_1, \dots, x_n)}_{\text{homogeneous deg } d \text{ symm fns.}}$$

Ex: $\underbrace{3m_{(2,2)} + 2m_{(3,1)}}_{\text{in } \Lambda^{(4)}} + \underbrace{4m_{(1,1,1)}}_{\text{in } \Lambda^{(3)}} + \underbrace{17}_{\text{in } \Lambda^{(0)}}$.

Each $\Lambda_{\mathbb{Q}}^{(d)}(x_1, \ldots, x_n)$ finite dimensional vector space:

basis $\{m_\lambda : \lambda \vdash d,\ \lambda \text{ at most } n \text{ parts}\}$

$$\dim = |\text{Basis}| = \sum_{k=1}^{n} p(d,k)$$

<u>Can clean up this alg.</u> by extending to ∞ many vars:

Zero maps:

$$\Lambda_{\mathbb{Q}}(x_1, \ldots, x_n) \xrightarrow{x_n \mapsto 0} \Lambda_{\mathbb{Q}}(x_1, \ldots, x_{n-1})$$

Ex: $\Lambda_{\mathbb{Q}}(x_1, x_2, x_3) \xrightarrow{x_3 = 0} \Lambda_{\mathbb{Q}}(x_1, x_2)$

$$\underbrace{m_{(3,1)} + 3m_{(2,1,1)} + 2m_{(4)}}_{\text{at most 3 parts}} \longrightarrow \underbrace{m_{(3,1)} + 2m_{(4)}}_{\text{remembers those w/ } \leq 2 \text{ parts.}}$$

So, partial compatibility:

$$\cdots \longrightarrow \Lambda(x_1, x_2, x_3) \longrightarrow \Lambda(x_1, x_2) \longrightarrow \Lambda(x_1)$$

<u>Inverse limit</u> of this chain is $\Lambda(x_1, x_2, x_3, \ldots)$

In down to earth terms:

<u>Def</u>: A <u>symm. function</u> in $x_1, x_2, x_3, \ldots$ is a bounded-deg sum of monomials invariant under any transposition $x_i \leftrightarrow x_{i+1}$.

<u>Ex</u>: $f = m_{(2,1)} + 3m_{(4)} = x_1^2 x_2 + x_1^2 x_3 + \cdots + \underset{\substack{\uparrow \\ \text{all } i,j}}{x_i^2 x_j} + \cdots$

$$+ 3x_1^4 + 3x_2^4 + 3x_3^4 + \cdots$$

Now have

$$\Lambda_{\mathbb{Q}} = \{\text{symm fns over } \mathbb{Q}\}$$
$$\cong \bigoplus \underset{\substack{\uparrow \\ \text{deg } d \text{ homog.}}}{\Lambda_{\mathbb{Q}}^{(d)}}$$

and $\dim \Lambda_{\mathbb{Q}}^{(d)} = p(d)$.

Basis: $\{m_\lambda : \lambda \text{ partition}\}$

Note: all of the above goes through for Λ_R, R any ring (not just $\mathbb{Q}$).