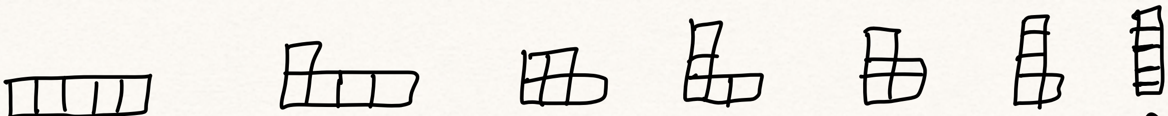
Partition bijections

Recall: $p(n) = \#$ partitions of n:

$\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, $\sum \lambda_i = n$

Young diagrams for $n=5$:



(5) $(4,1)$ $(3,2)$ $(3,1,1)$ $(2,2,1)$ $(2,1,1,1)$ $(1,1,1,1,1)$

So $p(5) = 7$.

Write $\lambda \vdash n$ for "λ is a partition of n".

Recall: $$\sum_{n=0}^{\infty} p(n)\, x^n = \frac{1}{(1-x)} \frac{1}{(1-x^2)} \frac{1}{(1-x^3)} \cdots$$

$Q(n) = \#$ partitions of n into <u>distinct</u> parts: $Q(5) = 3$

$$\sum Q(n)\, x^n = (1+x)(1+x^2)(1+x^3)\cdots$$

Lemma: Let $O(n) = \#$ partitions of n into odd parts. Then $Q(n) = O(n)$.

Pf 1: Gen fns: want to show

$$\frac{1}{(1-x)} \frac{1}{(1-x^3)} \frac{1}{(1-x^5)} \cdots = (1+x)(1+x^2)(1+x^3)\cdots$$

We show $(1-x)(1-x^3)\cdots(1-x^{2n-1}) \cdot (1+x)(1+x^2)(1+x^3)\cdots(1+x^n) \to 1$ as $n \to \infty$

Expands as:

$$(1-x)(1+x)(1+x^2)(1-x^3)(1+x^3)(1+x^4)\dots$$

$$(1-x^2)$$

$$(1-x^4) \qquad (1-x^6)\,(1+x^4)\dots$$

$$(1-x^8) \quad \text{etc.}$$

So the larger n gets, the larger the min x degree after x^0 gets

$\Rightarrow$ it converges to 1.

Bijective proof:

$n=11$

Distinct	Odd
5+3+2+1	5+3+1+1+1
5+4+2	5+1+1+1+1+1+1
6+5	5+3+3
6+4+1	3+3+1+1+1+1+1
6+3+2	3+3+3+1+1
7+4	7+1+1+1+1
7+3+1	7+3+1
8+3	3+1+1+1+1+1+1+1+1
	1+1+1+...+1
8+2+1	9+1+1
9+2	5+5+1
10+1	
11	11

Another ex: Partitions into distinct odd parts
$\leftrightarrow$ self-conjugate partitions

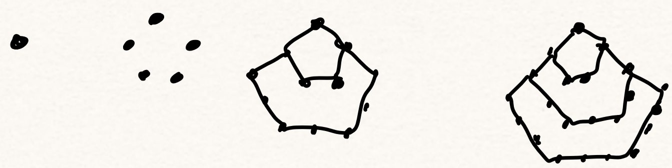
Computing $p(n)$: Euler's recursion

Pentagonal number: # dots in a pentagon w/ n dots on a side

Triangular #s:

$\frac{n(n+1)}{2}$: 1 3 6 10 ...

Pentagonal:



1 5 12 21 ...

($k=$ 1 2 3 4 ...)

Claim: k-th pentagonal number is $\boxed{\frac{k(3k-1)}{2}}$.

$k=1$: 1 ✓

Induct: add $3(k-1)+1$ to $(k-1)$-st pentagonal #:

$$\frac{(k-1)(3k-4)}{2} + 3(k-1)+1 = \frac{(k-1)(3k+2)+2}{2}$$

$$= \frac{3k^2-k}{2} = \frac{k(3k-1)}{2}. \quad ✓$$

Extend pentagonal #s to $-k$'s:

$$\frac{(-k)(3(-k)-1)}{2} = \frac{k(3k+1)}{2}$$

So a general pentagonal number is

$$\frac{k(3k\pm1)}{2} \quad \text{for } k\geq 0.$$

$$\frac{3\cdot 10}{2}$$

$k=0$	1	-1	2	-2	3	-3	
0	1	2	5	7	12	15	$\ldots$

<u>Thm:</u> $$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} p\left(n - \frac{k(3k-1)}{2}\right) + (-1)^{k+1} p\left(n - \frac{k(3k+1)}{2}\right)$$

i.e. $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \ldots$

<u>Ex:</u> $$\begin{aligned} p(5) &= p(4) + p(3) - p(0) \\ &= 5 + 3 - 1 \\ &= 7 \end{aligned}$$

$(p(0)=1)$
$(p(-n)=0)$

First show:

<u>Euler's Pentagonal Number Theorem</u>:

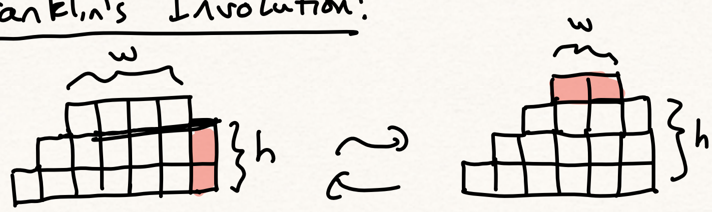
$$(1-x)(1-x^2)(1-x^3)\cdots = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right).$$

Pf: Let $Q(n,k)$ = # partitions of n into k distinct parts.

Then $(1-x)(1-x^2)(1-x^3)\cdots = \sum_{n=0}^{\infty}\Big(\underbrace{\sum Q(n,2k)}_{\text{even \# parts}} - \underbrace{\sum Q(n,2k+1)}_{\text{odd \# parts}}\Big)x^n$

Need a sign-reversing involution on partitions into distinct parts that pairs all, except when n is a pentagonal #.

Franklin's Involution:

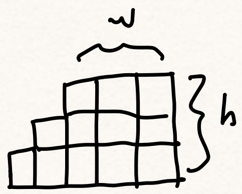


(draw $(7,6,4)$ as a shifted partition - shift i-th part over by i.)

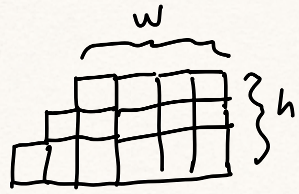
Drawing the Young diagram as a shifted partition (b/c distinct parts), let h be height of last column and w width of top row.

If $h \le w-1$, pick up the last col and make it a row on top. Otherwise, pick up top row and make it a rightmost column.

The only times this involution is not well-defined is:



and



If $k=h$, have $k^2 + \frac{k(k-1)}{2} = \frac{k(3k-1)}{2}$ boxes

If $k=h$, have $k^2 + \frac{k(k+1)}{2} = \frac{k(3k+1)}{2}$ boxes.

This completes the proof. $\square$

Now to show recursion:

$$\left(\sum p(n)x^n\right)(1-x)(1-x^2)\cdots = 1$$

$$\left(\sum p(n)\, x^n\right)\left(1+\sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{k(3k+1)}{2}} + x^{k(3k-1)/2}\right)\right) = 1$$

$$\sum\left(p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots\right) = 1$$

$\Rightarrow$ recursion holds for $n \geq 1$, and $p(0) = 1$.

QED