Möbius functions

Recall: $f: \mathrm{Int}(P) \to \mathbb{C}$ invertible iff $f(x,x) \neq 0$ for all x.

Recall: $\zeta(x,y) = 1$ for all $x \le y$

Def: The Möbius function of a poset P is $\boxed{\mu_P := \zeta_P^{-1}}$

Computing μ: $\mu \cdot \zeta = \delta \Rightarrow$

$$\sum_{x \le z \le y} \mu(x,z) \cdot \zeta(z,y) = \delta(x,y) = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$$

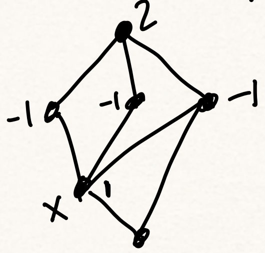
$\Rightarrow$ ① $\mu(x,x) = 1$

② $\sum_{x \le z \le y} \mu(x,z) = 0$ for $x < y$

② can be written as a recursion:

$$\mu(x,y) = -\sum_{x \le z < y} \mu(x,z)$$

Ex: Let's compute $\mu(x,z)$ for all $z > x$:



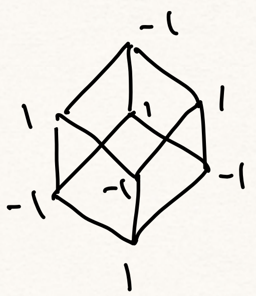
Sums of #s on any interval starting from x are 0.

Def: In a poset w/ $\hat{0}$, the <u>Möbius number</u> of an elt $z \in P$ is $\mu(\hat{0}, z)$.

Sometimes write $\mu(z) = \mu(\hat{0}, z)$ and call it the Möbius function on the poset.

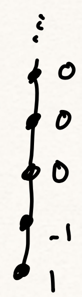
Def: If P has $\hat{0}$ and $\hat{1}$, its Möbius number is $\mu(\hat{0}, \hat{1})$.

Ex: Möbius function of Boolean lattice:

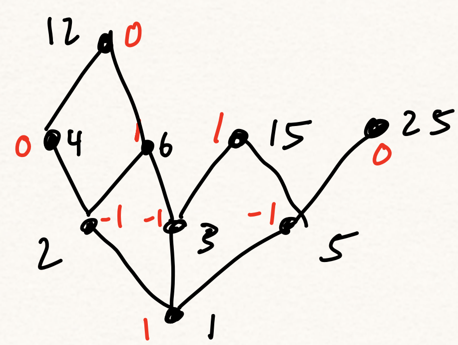


$$\mu(S) = (-1)^{|S|}.$$

Ex:



Ex:



$$\mu(n) = \begin{cases} 0 & \text{if } p^2 \mid n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 \cdots p_r \end{cases}$$

in divisor lattice

Hwk: Young's lattice.

Möbius Inversion

Statement: $\boxed{f \cdot \zeta = g \iff f = g \cdot \mu}$

Usual (more specific) setting:

Thm: Let P be a locally finite poset with a $\hat{0}$. Let $f, g : P \to \mathbb{C}$.

Then $g(x) = \sum_{y \leq x} f(y) \quad \forall x \in P$

$\iff f(x) = \sum_{y \leq x} g(y)\mu(y, x) \quad \forall x \in P.$

Pf: Define interval functions

$$\tilde{f}(\hat{0}, y) = f(y)$$
$$\tilde{g}(\hat{0}, y) = g(y)$$

and $\tilde{f}(x, y) = \tilde{g}(x, y) = 0$ for $x \neq \hat{0}$.

Then the condition

$(*)$ $g(x) = \sum_{y \leq x} f(y)$ is equivalent to

$$\tilde{g}(\hat{0}, x) = \sum_{\hat{0} \le y \le x} \tilde{f}(\hat{0}, y)\, \underbrace{\zeta(y, x)}_{=1}$$

$$= \tilde{f}\zeta(\hat{0}, x)$$

and we have, for $a \neq \hat{0}$,

$$\tilde{g}(a, b) = 0 = \sum_{a \le c \le b} \underbrace{\tilde{f}(a, c)}_{=0}\, \zeta(c, b)$$

so $(*) \iff \tilde{g} = \tilde{f}\zeta$

$$\iff \tilde{f} = \tilde{g}\zeta$$

$$\iff \tilde{f}(\hat{0}, x) = \sum_{y \le x} g(\hat{0}, y)\mu(y, x) \quad \text{by same logic}$$

$$\iff f(x) = \sum_{y \le x} g(y)\mu(y, x)$$

$\square$

Applications:

① $P = \text{chain } (\mathbb{Z}_{\geq 0}, \leq)$

$$\mu(i,j) = \begin{cases} 1 & i = j \\ -1 & i+1 = j \\ 0 & \text{else} \end{cases}$$

so $g(n) = \sum_{i=0}^{n} f(i)$

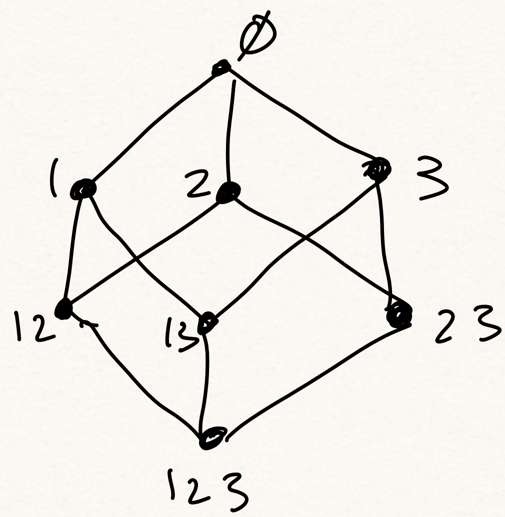
$\iff f(n) = g(n) - g(n-1)$

(discrete derivative)

② Inclusion-exclusion:

$P = B_n^* \rightsquigarrow$ dual poset of B_n

$= \{\text{subsets of } [n], \supseteq\}$



Fix sets $A_1, \ldots, A_n$

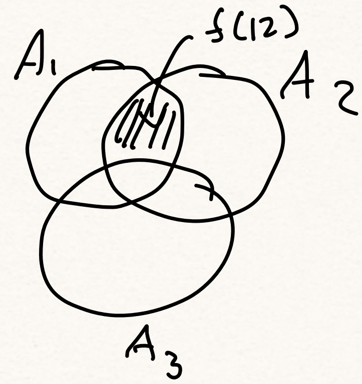
and define

$$g(S) = \left| \bigcap_{i \in S} A_i \right|$$

for $S \subseteq [n]$

$f(S) = \#$ elts in $\bigcap_{i \in S} A_i$ that are

not in any intersection contained in it.



Then

$$g(S) = \sum_{\substack{S' \supseteq S \\ (S' \leq_{B^*} S)}} f(S)$$

So by Möbius inversion:

$$f(S) = \sum_{T \supseteq S} g(T)\,\mu(T, S)$$

In particular:

$$0 = f(\emptyset) = \sum_{T \subseteq [n]} g(T)\,\mu(T, \emptyset) \qquad (**)$$

Lemma: In B_n, $\mu(\hat{0}, T) = (-1)^{|T|}$.

(equiv: In B_n^*, $\mu(T, \hat{1}) = (-1)^{|T|}$)

Pf: Induction on $|T|$; assume this is true for all S w/ $|S| < |T|$.

Then $\mu(\hat{0},T) = -\sum_{\substack{S\subseteq T\\ S\neq T}} \mu(\hat{0},S)$

$$= -\sum_{\substack{S\subseteq T\\ S\neq T}} (-1)^{|S|}$$

$$= -\sum_{s=0}^{|T|-1} (-1)^s \binom{|T|}{s}$$

$$= -\left(\left(\underset{\substack{\downarrow\\ 0}}{\sum_{s=0}^{|T|} (-1)^s \binom{|T|}{s}}\right) - (-1)^T\right)$$

$$= (-1)^T \qquad \square$$

So eqn $(**)$ becomes

$$0 = f(\emptyset) = \sum_{T\subseteq[n]} g(T)\mu(T,\emptyset) = \sum_{T\subseteq[n]} g(T)(-1)^T$$

which means $\sum_{T\subseteq[n]} \left|\bigcap_{i\in T} A_i\right| (-1)^T = 0$

which is inclusion-exclusion.

③ <u>Divisor lattice</u> $(\mathbb{Z}_+, |)$

Note: interval $[m, n]$ for $m|n$

isomorphic to $[\hat{0}, n/m]$

so $\mu(m,n) = \mu(\hat{0}, \frac{n}{m}) = \mu(\frac{n}{m})$

$$= \begin{cases} 0 & p^2 \mid \frac{n}{m} \text{ some prime } p \\ (-1)^r & \frac{n}{m} = p_1 \cdots p_r \end{cases}.$$

<u>Möbius inversion:</u>

Suppose $f, g : \mathbb{N} \to \mathbb{C}$.

then $$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)$$

<u>Ex</u>: $\sigma(n)$ = sum of divisors of n

$$= \sum_{d|n} d$$

$$\Rightarrow \quad n = \sum_{d|n} \sigma(d)\mu\left(\frac{n}{d}\right)$$

Ex: $\phi(n) = \#$ numbers less than n relatively prime to n.

Lemma: $\displaystyle\sum_{d|n} \phi(d) = n.$

Pf: Let $m < n$, $d = \gcd(m,n)$. Then $\frac{m}{d}$ is relatively prime to $\frac{n}{d}$, so is counted in $\phi(\frac{n}{d})$. We therefore have

$$\sum_{d|n} \phi(d) = \sum_{d|n} \phi\left(\frac{n}{d}\right) = n. \qquad \square$$

Cor: $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$

where $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$

Pf: Apply Möbius inversion to the Lemma:

$$\phi(n) = \sum_{d|n} d\,\mu\left(\frac{n}{d}\right) = \sum_{d|n} \frac{n}{d}\mu(d)$$

$$= \sum_{d = p_{i_1} \cdots p_{i_k}} \frac{n}{p_{i_1} \cdots p_{i_k}} (-1)^k$$

which is equal to the factorization.

QED

Ex: $\phi(60) = 60 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 16$

$$\left.\begin{array}{l} 1, 7, 11, 13, 17, 19, 23, 29 \\ 59, 53, 49, 47, 43, 41, 37, 31 \end{array}\right\} \text{rel prime to } 60$$

Ex: of inclusion-exclusion

D_n = # derangements of n

$$= \underbrace{n!}_{\text{all perms}} - \underbrace{(n-1)!}_{\text{one elt in its place}} \cdot n + (n-2)!\binom{n}{2} - \cdots \pm (n-n)!\binom{n}{n}$$

$$= n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \cdots \pm \frac{n!}{n!}$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{n!}\right)$$

$$\Rightarrow \boxed{D_n \approx \frac{n!}{e}}$$