Subposets, intervals, order ideals

Def: An <u>induced</u> <u>subposet</u> of P is a subset of the elts of P along w/ <u>all</u> the relations in $\leq_P$ among them

Ex: The <u>interval</u> $[s,t] \subseteq P$ is the induced subposet on $\{u \in P : s \leq u \leq t\}$.

Def: An <u>order</u> <u>ideal</u> I is a downwards closed subposet of P: if $t \in I$, $s \leq t$, then $s \in I$.

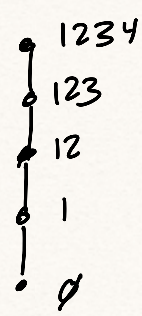
<u>Principal</u> <u>order ideal</u>: $(t) = \{s \in P : s \leq t\}$

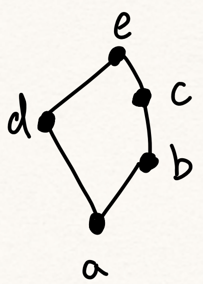
$J(P)$ = poset of order ideals under $\subseteq$

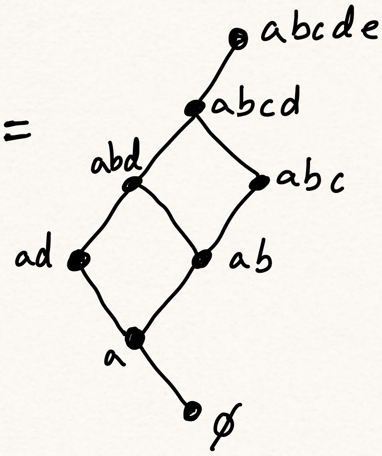
Exs of $J(P)$ construction

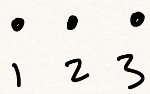
$P =$ 

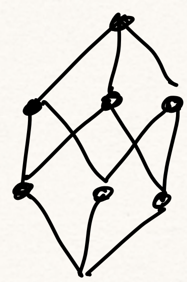
$\longrightarrow$

$J(P) =$ 

$P =$ 

$J(P) =$ 

$P =$ 

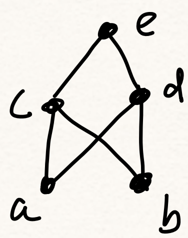
$J(P) =$ 

$J(P)$'s are "distributive lattices"

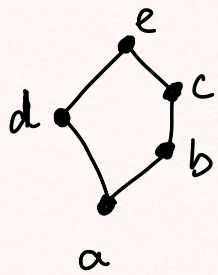
<u>Lattices</u>

<u>Def</u>: If $s,t \in P$, an <u>upper bound</u> is an elt $u \geq s,t$. <u>Least upper bound</u> if for any other upper bound u' of s,t, we have $u \leq u'$.

L.U.B. also called <u>join</u>, write $u = s \vee t$.

Ex: 

a and b have no join; three upper bounds, c, d, e.

Ex: 

$b \vee d = c \vee d = e$

Ex: In B_n, join is union.

Ex: In $(\mathbb{Z}_+, |)$, join is LCM

Def: lower bound of s, t: u s.t. $u \leq s, t$

u greatest lower bound or meet: if $u' \leq s, t$, $u' \leq u$.

$$u = s \wedge t$$

Ex: In B_n, meet is intersection

Ex: In $(\mathbb{Z}_+, |)$, meet is gcd

Def: P is a lattice if every pair of elts has a meet and a join.

Properties of Lattices

1. $\vee$, $\wedge$ are associative and commutative
2. Absorption: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$
3. $x \wedge y = x \iff x \vee y = y \iff x \leq y$
4. Finite lattices have a $\hat{0}$ and a $\hat{1}$.

meet of all elts, min

join, max

Def: A meet-semilattice (resp. join-semilattice) is a poset w/ meets (resp. joins)

Prop: A finite meet-semilattice w/ a $\hat{1}$ is a lattice.

Pf: Let s, t be elts. Then s, t have an upper bound b/c of $\hat{1}$. Take the meet of all the upper bounds of s, t:

$$u := \bigwedge_{v \geq s,t} v$$

Then since s is a lower bound of the v's, have $u \geq s$. Similarly $u \geq t$. So u is a common upper bound.

For any other $u' \geq s, t$, have $u' \wedge u = u$ so $u' \geq u$. Hence $u = s \vee t$. So joins exist.

QED.

(Converse: A finite join-semilattice w/ a $\hat{0}$ is a lattice).

Types of lattices

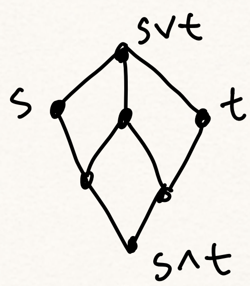
① Upper semimodular: Finite graded lattice w/

$$rk(x)+rk(y) \geq rk(x \wedge y)+rk(x \vee y)$$

Modular if equality hold

Ex: Sets, vector spaces modular

Semimodular but not modular:



Prop: Finite lattice L is modular iff

$\forall x,y,z \in L$ s.t. $x \leq z$,

$$x \vee (y \wedge z) = (x \vee y) \wedge z$$

(will not prove)

② **Atomic**: An *atom* is an elt that covers $\hat{0}$.

A lattice is *atomic* if every elt is a join of atoms.

Def: A lattice is *geometric* if it is finite, semimodular, and atomic. (SOZ!)

③ **Distributive**: if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Note: Any lattice of sets w/ intersection and union is distributive.

Claim: For any finite poset P, $J(P)$ is a finite distributive lattice.

Pf: Finite by def.

Lattice: If I, I' are order ideals, so are their union and intersection, so meets and joins exist.

Distributive: Intersection and union distribute across each other. $\square$

Fundamental Thm of Finite Distributive Lattices

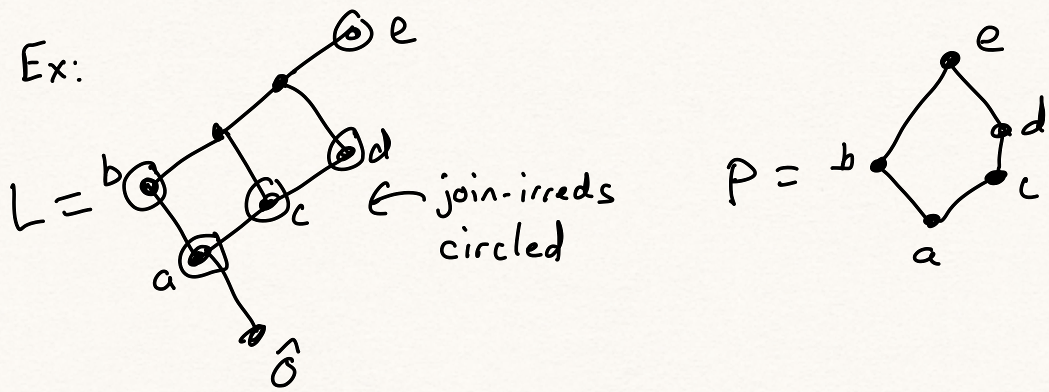
Converse holds: A finite lattice L is distributive iff it is $J(P)$ for some P.

Pf: Let L a finite distr. lattice.

P = subposet of L induced by the nonzero ($\neq \hat{0}$) "join-irreducible" elements. [not the join of two distinct elts]

Claim: $L = J(P)$

Ex:



Pf: For $x \in L$ define $I_x = \{ y \in P : y \leq_L x \}$

I_x order ideal in P, so $I_x \in J(P)$.

Gives map

$$\varphi : L \to J(P)$$
$$x \mapsto I_x .$$

Sagan proves this is a bijection. (prop 5.3.6 and pag 155).