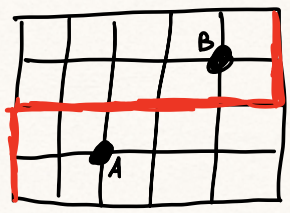
Posets

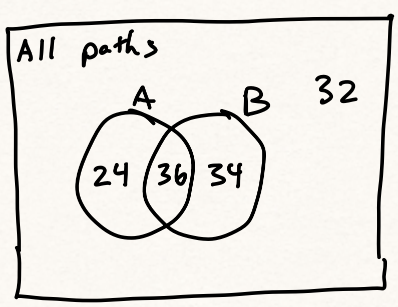
Warmup: How many up-right paths in the grid from lower left to upper right avoid both potholes?



● = pothole

- $\binom{9}{4} = 126$ total paths, including potholes
- $\binom{3}{1} \cdot \binom{6}{3} = 60$ paths through pothole A
- $\binom{7}{3} \cdot \binom{2}{1} = 70$ paths through pothole B
- $\binom{3}{1} \cdot \binom{4}{2} \cdot \binom{2}{1} = 3 \cdot 6 \cdot 2 = 36$ paths through <u>both</u> potholes

Venn diagram:



Inclusion-exclusion:

\# paths that involve a pothole =

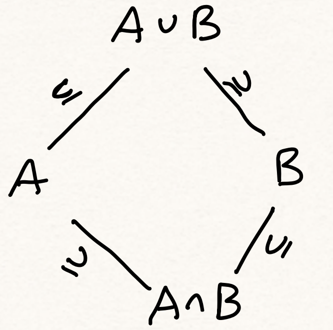
$60 + 70 - 36 = 94$

$\Rightarrow$ \# paths w/ no pothole

$= 126 - 94 = 32$

"Partially ordered set" – partial ordering is $\subseteq$

Poset:



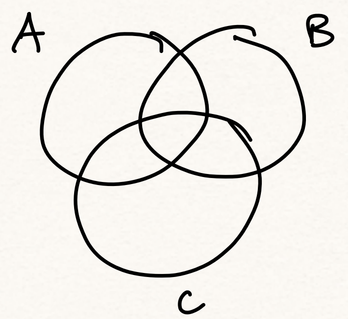
Inclusion-exclusion:

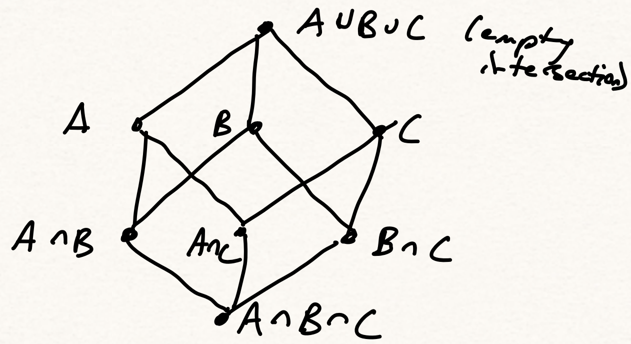
$$|A \cup B| = |A| + |B| - |A \cap B|$$

i.e.

$$|A \cup B| - |A| - |B| + |A \cap B|$$

Inc.-exc. w/ 3 sets:





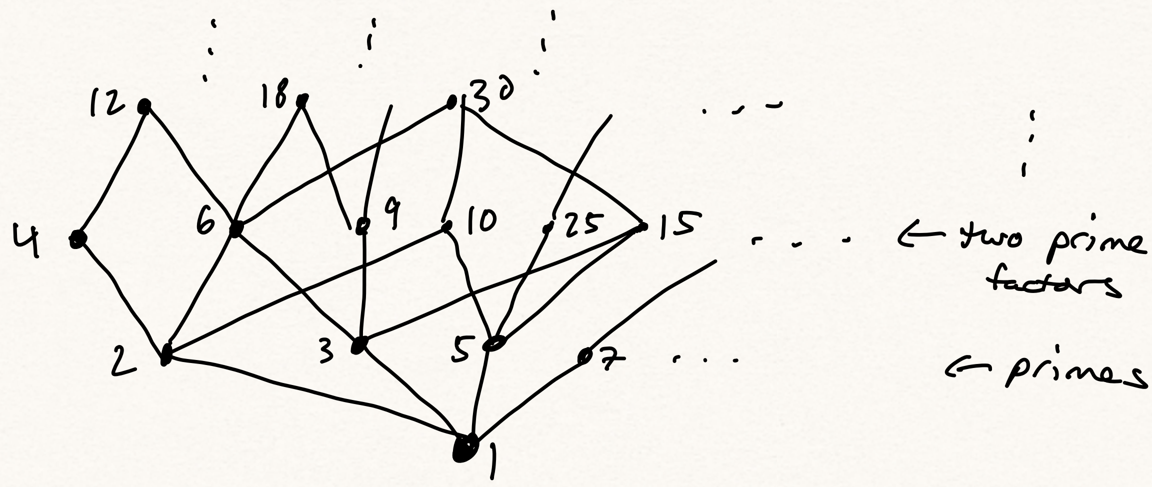
$$|A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| = 0$$

<u>Another ex:</u> from number theory!

$$\mu(n) = \begin{cases} 0 & \text{if } p^2 \mid n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 \cdots p_r \quad \text{(product of distinct primes)} \end{cases}$$

Then if $f(n) = \sum_{d \mid n} g(d)$, then $g(n) = \sum_{d \mid n} \mu(d) f(d)$!

Poset of numbers where the relation is divisibility:



<u>Def</u>: A <u>poset</u>, or <u>partially ordered set</u>, is a pair $(P, \leq)$ s.t. P - set, $\leq$ relation on P (subset of $P \times P$) s.t.

1. <u>Reflexive</u>: $x \leq x \quad \forall x \in P$
2. <u>Antisymmetric</u>: $x \leq y,\ y \leq x \Rightarrow y = x$
3. <u>Transitive</u>: $x \leq y,\ y \leq z \Rightarrow x \leq z$.

Write $s < t$ if $s \leq t$, $s \neq t$.

<u>Ex</u>: $(\mathbb{Z}, \leq)$

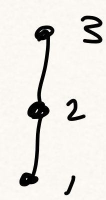
$(\mathbb{Z}_+, \leq)$

$(\mathbb{Z}, \geq)$

$(\mathbb{R}, \leq)$

$(\{1, 2, 3\}, \leq)$

↑ finite poset



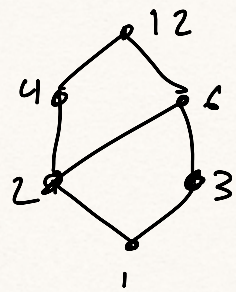
$(\mathbb{Z}_+, \text{divides})$

↑

divides symbol: $|$

$a|b$

$(\{1,2,3,4,6,12\}, |) \rightarrow$



$(B_n, \subseteq)$

↑

boolean poset: $B_n = \{\text{subsets of } 1,\ldots,n\}$

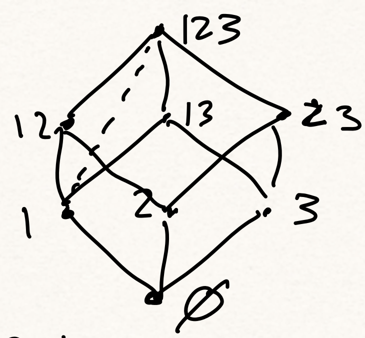
$(B_3, \supseteq)$ - is intersection poset of 3 sets! (above)

Hasse Diagram:

Def: t covers s if $s < t$ and no $u \in P$ satisfies $s < u < t$. Write $s \lessdot t$.

Covering relations "generate" all $\leq$ relations by transitivity; only draw coverings as edges, upwards.

Ex: B_3:



don't draw dashed edge.

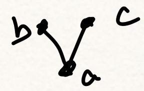
Def: An <u>isomorphism</u> of posets is a bijection $\varphi: P \to Q$ s.t.

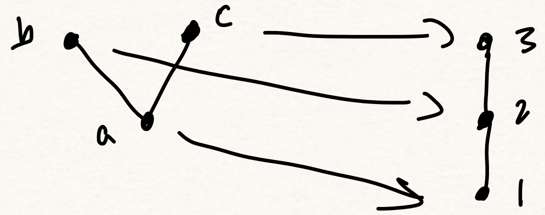
$$s \leq_P t \iff \varphi(s) \leq_Q \varphi(t)$$

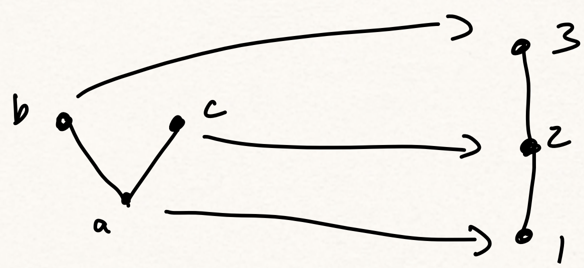
Def: <u>Linear extension</u> of a finite poset P to a total ordering is a bijective map

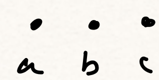
$$\varphi: P \to ([n], \leq) \qquad (\text{where } n = |P|)$$

s.t. $s \leq_P t \Rightarrow \varphi(s) \leq \varphi(t)$

Ex: 2 lin exts of  :





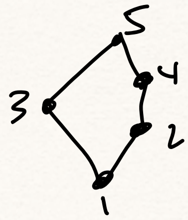
Ex: How many lin. exts of  are there?

Note: existence of lin. ext. means Hasse diagram can be drawn upwards.

Lemma: Every finite poset has a linear ext.

Pf: Induct on $|P|$. Follow a chain $a_1 \leq a_2 \leq \cdots$ to find a maximal elt b at which it terminates. Set $\varphi(b) = n$, take a linear ext. of remaining poset to $1, \ldots, n-1$; this is compatible. $\square$

Def: Maximal chain of P: A chain not contained in a larger chain:



1,2,4,5 and 1,3,5 both max. chains

Saturated chain: can't add elts between others to extend chain.

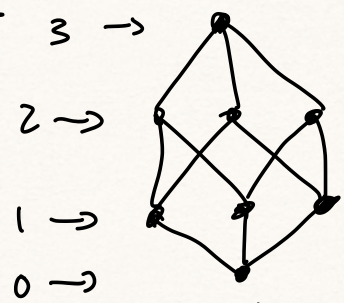
Ex: In $(\mathbb{Z}_+ \cup \{\infty\}, \leq)$, $\mathbb{Z}_+$ is saturated but not maximal. $\{1,2,3\}$ also saturated.

Length of a chain: $\#C - 1$ = # edges in Hasse diagram.

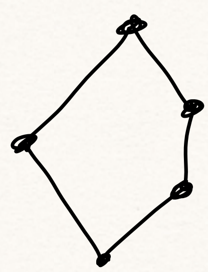
Def: The rank of a finite poset is its max chain length.

Def: P is graded of rank n if every maximal chain has the same length n.

Ex:

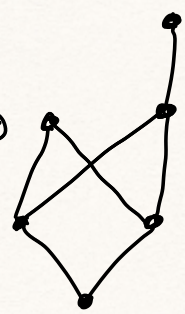


graded, rank 3

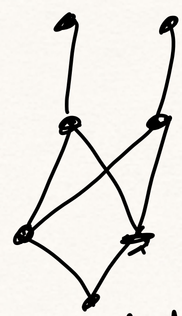


not graded

is this rank 2 or 3??



← not graded



graded

Rank of an elt x of a graded poset P is length of any largest chain w/ x at top.

<u>Rank generating function</u>: of graded poset P:

$$P_q = \sum_{p \in P} q^{rk(p)}$$

<u>Exs</u>:

- $[n]_q = [n]_q$
- $(B_n)_q = (1+q)^n$
- $(Y_{k,n-k})_q = \binom{n}{k}_q$ ⟶ poset of partitions in $k \times (n-k)$ box under containment
- <u>Bruhat order</u> ⟶ $(n)_q!$