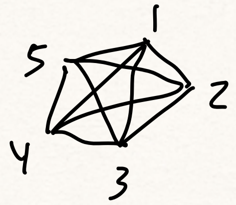
Ramsey Theory (Lectures 22–23)

Def: Complete graph K_n: $V = [n]$, all edges (i,j)

K_5:



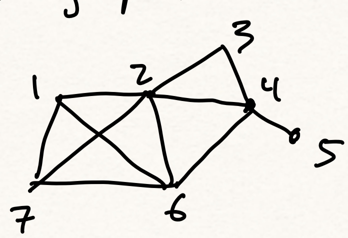
Q: How many edges? $\binom{n}{2}$

Def: A k-clique in a graph G is a subset of k of the vertices, every pair of which is joined by an edge.

(K_k-subgraph)

(only considering simple undirected graphs – no loops, mult. edges)

Ex:



- 4-clique: 1, 2, 6, 7
- 3-cliques: triangles
- All edges are 2-cliques.

Edge colorings: A d-coloring of the edges of G is a coloring (labeling) of each edge by one of d colors

Note: A 2-coloring of K_n is a graph and its "edge complement"

Def: (Ramsey numbers)

$R(a_1, \dots, a_d)$ = smallest n s.t. any d-colored K_n contains either:

- An a_1-clique of color 1 OR
- An a_2-clique of color 2 OR

⋮

- An a_d-clique of color d.

Ex: $R(r,s)$ = smallest n s.t. any 2-coloring of K_n (red/blue) contains either a red K_r or blue K_s.

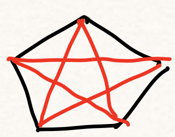
Ex: $R(1,s) = 1$

$R(2,s) = s$

$R(3,3) = ?$

Claim: $R(3,3) = 6$.

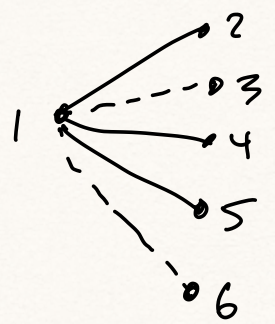
- $R(3,3) > 5$:



- $R(3,3) \leq 6$:

Consider edges 1-2, 1-3, 1-4, 1-5, 1-6

Some 3 are the same color



(say 12, 14, 15 same color)

Then 25, 24, 45 all the dashed color to avoid a solid triangle ⇒ there's a dashed triangle. QED

Ex: $R(3,3,2) = 6$ (why?)

Ex: $R(3,3,3) = 17$

Ramsey #s hard to compute in general!

$R(r,s)$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8	9	10
3			6	9	14	18	23	28	36	
4				18	25	36-41	49-61			
5					43-48		⋱			?
6										
7							205-540			

(rows: r; columns: s)

Ramsey's Theorem

$R(r,s)$ is always finite (and in fact $R(a_1, \ldots, a_d)$ is always finite)

Pf: Induct on $r+s$ (or on $a_1 + \ldots + a_d$; we do the proof for $d=2$ and it generalizes).

- $R(1,1)=1$ (base case)
- Claim: $R(r,s) \leq R(r-1,s) + R(r,s-1)$

Pf of claim: Let $n = R(r-1,s)$, $m = R(r,s-1)$

Consider a 2-coloring of

K_{n+m}.

Consider all edges from 1; $n+m-1$ of them.

Either $\geq n$ of them are red or $\geq m$ of them are blue.

Case 1: $\geq n$ edges from 1 are red.

Then among the vertices $v_1, \ldots, v_n$ conn. to 1 by a red edge,

either a red $(r-1)$-clique or a blue s-clique since $n = R(r-1,s)$.

A red $(r-1)$-clique makes a red r-clique w/ 1, so done.

Case 2: $\geq m$ edges from 1 are blue. Same argument.

The thm now follows from the Claim and induction. $\square$

Lower bound, probabilistic method

Prob. Method: If something happens with positive probability, then it sometimes happens (existence)

Thm: If $k \geq 3$ and $\binom{n}{k} \cdot 2 < 2^{\binom{k}{2}}$, then $R(k,k) > n$. As a result, $R(k,k) > \lfloor 2^{k/2} \rfloor$ for $k \geq 3$.

Pf: Consider a randomly selected 2-coloring of K_n.

For any subset of k of the vertices, prob. that it's a K_k is

$$\frac{2}{2^{\binom{k}{2}}}.$$

So, prob. that at least one monochr. k-clique occurs is bounded above by

$$\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}}.$$

But by the assumed inequality we have

$$\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}} < 1$$

So, w/ pos. probability, <u>no</u> monochrom. k-clique occurs.

$\Rightarrow R(k,k) > n.$

<u>Deriving inequality</u>: Need to show

$\binom{n}{k} \cdot 2 < 2^{\binom{k}{2}}$ for $n = \lfloor 2^{k/2} \rfloor$:

$$\binom{\lfloor 2^{k/2} \rfloor}{k} \cdot \frac{2}{2^{\binom{k}{2}}} < \frac{(2^{k/2})^k}{k!} \cdot \frac{2}{2^{k^2/2 - k/2}}$$

$$= \frac{2 \cdot 2^{k/2}}{k!} < 1 \quad \text{for } k \geq 3.$$

QED