Proof of Cayley using Matrix-Tree

First, cor: undirected matrix-tree thm:

Thm: # Spanning trees in undir. graph G $= \det(\tilde{L}(G))$

where $L(G) = \begin{pmatrix} \deg v_1 & & \\ & \ddots & \\ & & \deg v_n \end{pmatrix} - A(G)$

$a_{ij} = $ # edges $i - j$

and $\tilde{L}$ formed by deleting k^{th} row, col for any k.

Cor 2: If $\lambda_1, \dots, \lambda_n$ eigenvalues of $L(G)$ w/ $\lambda_n = 0$,

spanning trees is $\frac{1}{n} \lambda_1 \lambda_2 \cdots \lambda_{n-1}$ (w/ multiplicity)

Pf: Recall eigenvalue is a value λ s.t. $\exists$ vector v (eigenvector), $L(G) \cdot v = \lambda v$.

- Eigenvectors span (eigenbasis)
- $\det(L - xI) = (-1)^n (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$

In this case, const. term 0.

Add all rows of $L - xI$ to final row; this kills const. terms, last row all div. by $-x$.

Let $N(x)$ = matrix formed by dividing bot. row by $-x$:

$$\det(L - xI) = -x \det N(x)$$

$$\Rightarrow \quad \frac{\det(L - xI)}{-x} = \det N(x)$$

$$\Rightarrow \quad (-1)^{n-1}(x-\lambda_1)\cdots(x-\lambda_{n-1}) = \det N(x)$$

$$\Rightarrow \quad \lambda_1 \cdots \lambda_{n-1} = \det(N(0)).$$

Add cols of $N(0)$ to last column: $\left(\begin{array}{c|c} \tilde{N}(0) & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0\ 0\ \cdots\ 0 & n \end{array}\right)$

$$\Rightarrow \quad \det N(0) = \det(\tilde{N}(0)) \cdot n$$

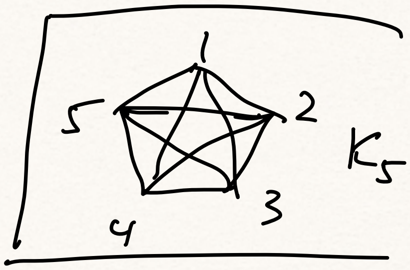
$$= \det(\tilde{L}) \cdot n$$

$$= \#\text{sp.trees} \cdot n$$

$$\Rightarrow \quad \#\text{ sp. trees} = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}.$$

<u>Proof of Cayley's Thm</u>: # trees on n labeled vertices is n^{n-2}.

Same as counting spanning trees of complete graph K_n



$$L = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

<u>Eigens:</u> $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ has eigenvalue 0,

the vectors $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ etc have eigen 5

$\Rightarrow \quad \frac{1}{5}(5^4) = \boxed{5^3}$ $\qquad$ (n^{n-2} in general)

<u>Alternatively:</u>

$$\det(\tilde{L}) = \det\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} = \det\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 \\ -1 & -1 & 4 & -5 \\ -1 & -1 & -1 & 5 \end{pmatrix}$$

$$= \det\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 \\ -2 & -2 & 3 & 0 \\ -1 & -1 & -1 & 5 \end{pmatrix} = \det\begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -5 & 0 \\ -2 & -2 & 5 & 0 \\ -1 & -1 & 0 & 5 \end{pmatrix}$$

$$= \det\begin{pmatrix} 4 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ -2 & -2 & 5 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 4 & -5 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ -2 & 0 & 5 & 0 \\ -1 & 0 & 0 & 5 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ -2 & 0 & 5 & 0 \\ -1 & 0 & 0 & 5 \end{pmatrix} = 5^3.$$

QED.