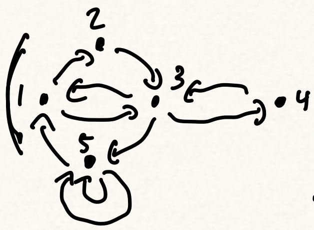
(Reminder: Do adj. matrices and counting sp. trees first)

Matrix-Tree Thm

Def: The *Laplacian* matrix of a digraph D is

$$\mathcal{L}(D) = \begin{pmatrix} \text{outdeg}(v_1) & & & 0 \\ & \text{outdeg}(v_2) & & \\ & 0 & \ddots & \\ & & & \text{outdeg}(v_n) \end{pmatrix} - A(D)$$

adjacency matrix.

Ex: $\mathcal{L}\Big($  $\Big) = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$

Deleted Laplacian: $\tilde{\mathcal{L}}_k(D)$: delete k^{th} row, col from $\mathcal{L}(D)$.

Ex: $\tilde{\mathcal{L}}_3(\text{above}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Thm: $\det(\tilde{\mathcal{L}}_k(D)) = \tau(D, v_k) = \#$ oriented sp. trees rooted at v_k.

In Ex: det = 2, 2 sp. trees rooted at 3.

Facts About Determinants

1. 2×2: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

2. $n\times n$: (def) $\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1\pi_1} \cdot a_{2\pi_2} \cdot \cdots \cdot a_{n\pi_n}$

3. Switching two rows or two cols negates det

4. Scaling a row or col by λ scales det by λ

5. Adding a row to another row doesn't change det

6. Multiplicative: $\det(AB) = (\det A)(\det B)$

7. $\det A \neq 0$ iff A invertible iff rows are independent, iff cols are indep.

8. $|\det A|$ = volume of parallelipiped spanned by col vectors.

9. $\det\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = \det A \cdot \det B$

10. $\det(A) = \sum a_{1i} \det(\hat{A}_i)(-1)^i$
   delete 1st row, ith col.

11. $\det\begin{pmatrix} \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \\ & \vdots & \\ \text{---} & v_n & \text{---} \end{pmatrix} + \det\begin{pmatrix} \text{---} & u & \text{---} \\ \text{---} & v_2 & \text{---} \\ & \vdots & \\ \text{---} & v_n & \text{---} \end{pmatrix} = \det\begin{pmatrix} \text{---} & v_1+u & \text{---} \\ \text{---} & v_2 & \text{---} \\ & \vdots & \\ \text{---} & v_n & \text{---} \end{pmatrix}$

Proof of Matrix-Tree

Induct on # edges of D.

Base cases

- If # edges $< n-1$, graph is disconnected.

$G \sqcup H = D$, $\quad v_k \in G \quad \Rightarrow$

$$\det(\tilde{L}(D)) = \det\left(\begin{array}{c|c} \tilde{L}(G) & 0 \\ \hline 0 & L(H) \end{array}\right)$$

$$= \det(\tilde{L}(G)) \cdot \underbrace{\det(L(H))}_{\substack{\| \\ 0}}$$

$= 0 \quad \checkmark$

$=$ #sp trees of disconn. graph.

- If # edges $= n-1$, if disconn. it's 0 as above. If connected, a tree.

  $\rightarrow$ If oriented towards v_k $(\tau = 1)$ than if we order vertices outwards from root, matrix is lower Δ, 1's on diagonal $\Rightarrow$ $\det = 1$ $\checkmark$

  $\rightarrow$ If not oriented towards v_k, some vertex has outdeg 0

  $\Rightarrow \det = 0$, no sp. trees. $\checkmark$

Induction step: Assume true for all #s of edges $< m$ $(m > n-1)$.

Can assume v_k has no out edges - this only affects row k, no sp. trees involve it.

So $\exists$ vertex $u \neq v_k$ w/ outdeg$(u) \geq 2$. Let $e_1, \ldots, e_r$ be out edges from u, consider

D_1 = delete e_1.

D_2 = delete $e_2, \ldots, e_r$.

Have

$$\tilde{\mathcal{L}}(D_1) = \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_u' & — \\ & \vdots & \\ — & v_n & — \end{pmatrix}, \quad \tilde{\mathcal{L}}(D_2) = \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_u'' & — \\ & \vdots & \\ — & v_n & — \end{pmatrix}$$

$v_u' + v_u'' = v_u$ from $\mathcal{L}(D)$

$$\Rightarrow \det \tilde{\mathcal{L}}(D) = \det(\tilde{\mathcal{L}}(D_1)) + \det(\tilde{\mathcal{L}}(D_2))$$

$$= \tau(D_1, v_k) + \tau(D_2, v_k) \quad \text{by induction}$$

$$= \tau(D, v_k).$$

QED