More q-analogs

Def: $\mathrm{exc}(\pi) = \#\{i : \pi_i > i\}$

S_3	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

Hint for proving des, exc are equidistr:
Bijection
$\varphi: S_n \to S_n$
by: write π in cycle notation w/ biggest # first in each cycle, max elts increasing.
$\varphi(\pi)$: drop parentheses

Ex: π $\qquad$ $\varphi(\pi)$

$(412)(63)(875) \longrightarrow 41263875$

Recall: $\binom{n}{k}_q = \sum_{w \in S_{0^k,1^{n-k}}} q^{\mathrm{inv}(w)} = \frac{(n)_q!}{(k)_q!\,(n-k)_q!}$

Lemma (q-Pascal):

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}$$

Pf: Any word either starts w/ 0 or 1, and these cases give the two terms on RHS.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 1+q & & 1 & & \\
 & 1 & & 1+q+q^2 & & 1+q+q^2 & & 1 & \\
1 & & 1+q+q^2+q^3 & & 1+q+2q^2+q^3+q^4 & & 1+q+q^2+q^3 & & 1
\end{array}
$$

symmetric,
unimodal coeffs (Hard!)

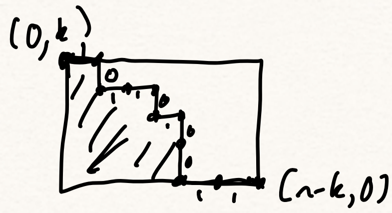
<u>q-Stirling</u>

$$S(n,k)_q = S(n-1,k-1)_q + (k)_q\, S(n-1,k)_q$$

<u>Another interp of $\binom{n}{k}_q$:</u>

$$\binom{n}{k}_q = \sum_{\substack{\text{paths } p:\\ (0,k)\to(n-k,0)}} q^{\text{area}(p)}$$

$$= \sum_{\lambda \subset \underbrace{\boxed{}}_{n-k}\, k} q^{|\lambda|}$$



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invs = boxes below path

Generating Functions

g.f. of sequence $a_0, a_1, a_2, \ldots$

is the formal power series

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots$$

Here x is just a formal symbol and we are going to define operations on formal power series that are really just operations on sequences.

Sums $$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

Products: $$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

Intuition: $$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

(Next hw: Show sums, products satisfy associativity, commut, distrib. etc)

$R[[x]]$ - ring of formal power series in x w/ coeffs in R. R can be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}[q]$, etc.

R is a ring too — elts can be added and multiplied, have negatives but not nec reciprocals.

Note: $\sum_{n=0}^{\infty} n!\, x^n$ well defined!

Generating function identities

$$(1+x+x^2+\cdots)(1-x+0x^2+0x^3+\cdots) = 1+0x+0x^2+\cdots$$

$$\Rightarrow \quad (1+x+x^2+\cdots)(1-x) = 1$$

write: $\quad 1+x+x^2+\cdots = \underbrace{\frac{1}{1-x}}$

"closed form" - expressed in terms of known or polynomial generating functions.

Ex: $1+2x+3x^2+4x^3+\cdots = \frac{1}{(1-x)^2}$

Differentiation

Def: $\frac{d}{dx}\left(\sum_{i=0}^{\infty} a_i x^i\right) = \text{"}\sum_{i=1}^{\infty} i a_i x^{i-1}\text{"} = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i$

Cleaner operation: $x\frac{d}{dx}$

$$x\frac{d}{dx}\left(\sum_{i=0}^{\infty} a_i x^i\right) = \sum_{i=1}^{\infty} i a_i x^i$$

↑ multiplies i-th term by i.

Thm: (Hw 5) $\frac{d}{dx}$ satisfies usual rules of differentiation.

- Sum rule, product rule, chain rule, etc

↓ need composition

Composition: If $F(x) = \sum_{n=0}^{\infty} f_n x^n$, $G(x) = \sum_{n=0}^{\infty} g_n x^n$,

$F \circ G(x)$ is only defined when $g_0 = 0$.

If so:

$$F \circ G(x) = \text{"}\sum_{i=0}^{\infty} f_i (g_1 x + g_2 x^2 + \dots)^i\text{"} = \sum_{n=0}^{\infty}\left(\sum_{\substack{1 \le k \le n \\ \sum a_i = n}} f_k g_{a_1} \cdots g_{a_k}\right) x^n$$

Monomial substitution (ex. of composition)

$F(3x^2)$.

Ex: Let $F(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$.

$$F(3x^2) = 1 + 3x^2 + 9x^4 + 27x^6 + \cdots = \frac{1}{1-3x^2}$$

Ex: What is:

$$1 - 2x + 4x^2 - 8x^3 + 16x^4 - \cdots ?$$

(Find a closed form)

Index shift: mult by x^r

$$x^r \cdot \sum_{i=0}^{\infty} a_i x^i = \sum_{i=r}^{\infty} a_{i-r} x^i$$

Consecutive sums: mult by $(1+x)$

Consecutive differences: mult by $(1-x)$

Partial sums: Mult by $\frac{1}{1-x}$:

$$\frac{1}{1-x}\left(\sum a_i x^i\right) = (1 + x + x^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots$$

Exponentials:

Def: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ **Note:** $0! = 1$

↑ also written $E(x)$

Def: Exponential g.f of $a_0, a_1, a_2, \ldots$ is

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

Some benefits:
- $\frac{d}{dx} \sum \frac{a_n}{n!} x^n = \sum \frac{a_{n+1}}{n!} x^n$

  is Index shift.
- Product: $\left(\sum \frac{a_n}{n!} x^n\right)\left(\sum \frac{b_n}{n!} x^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) x^n$

Using generating functions in combinatorics

- Binomial thm:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

- Choose w/ repeats:

$$\sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) x^k = \frac{1}{(1-x)^n}$$

- $\sum_{k=0}^{\infty} n^k x^k = \frac{1}{1-nx}$

- $\sum_{k=0}^{\infty} (n)_k x^k = 1 + nx + n(n-1)x^2 + n(n-1)(n-2)x^3 + \dots + n!x^n$

$$= 1 + nx\left(\sum_{k=0}^{\infty} (n-1)_k x^k\right)$$

$$A_n = 1 + nx A_{n-1}$$

Proving identities:

- Plug $x=-1$ into binomial thm:

$$\sum (-1)^k \binom{n}{k} = 0$$

- Plug $x=1$ into binom thm:

$$\sum \binom{n}{k} = 2^k$$

When can we plug in?

Thm: If $A(x) = B(x)$ as generating functions and A, B converge at $x=c$, then $A(c) = B(c)$.

Identity: $\frac{(1-x)^n}{(1-x)^n} = 1$

$$\Rightarrow \left(\sum_k (-1)^k \binom{n}{k} x^k\right)\left(\sum_k \left(\!\binom{n}{k}\!\right) x^k\right) = 1$$

$$\Rightarrow \sum_{j=0}^{k} (-1)^j \binom{n}{j} \left(\!\binom{n}{k-j}\!\right) = 0 \qquad \text{for } k>0$$

Finding explicit formulas for counting

Ex: How many binary seq. of length n have no two consecutive 1's?

Ans: Fibonacci F_{n+2}

len	0	1	2	3	4	
num	1	2	3	5	8	...

Fibonacci: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

Formula for n^{th} Fibonacci number?

Consider generating function:

$$F(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + F_5 x^5 + \cdots$$

$$= x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots$$

Step 1: use recursion to solve for $F(x)$.

Recall mult by x, x^2 does index shift:

$$\begin{aligned}
F(x) &= x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots \\
-\,xF(x) &= \phantom{x + {}} x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + \cdots \\
-\,x^2F(x) &= \phantom{x + x^2 + {}} x^3 + x^4 + 2x^5 + 3x^6 + \cdots
\end{aligned}$$

$(1-x-x^2)F(x) = x$

$\boxed{\underline{\text{Char. polynomial}}: 1-x-x^2}$

$$\Rightarrow \boxed{F(x) = \frac{x}{1-x-x^2}}$$

<u>Step 2</u>: Expand the g.f. a different way:

Want to factor denom as

$$1-x-x^2 = (1-ax)(1-bx)$$

$a+b=1$
$ab=-1$

$a(1-a)=-1$

$a-a^2+1=0$

$a^2-a-1=0$

$a = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = b$

$$F(x) = \frac{c}{1-ax} + \frac{d}{1-bx}$$

$c+d=0 \Rightarrow d=-c$

$$= \frac{c}{1-ax} - \frac{c}{1-bx}$$

$ca-cb=1$

$c(\sqrt{5})=1$

$c = 1/\sqrt{5}$

$a = \frac{1+\sqrt{5}}{2}$

$b = \frac{1-\sqrt{5}}{2}$

$$F(x) = \frac{1}{\sqrt{5}} \frac{1}{1-\left(\frac{1+\sqrt{5}}{2}\right)x} - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\left(\frac{1-\sqrt{5}}{2}\right)x}\right)$$

$$= \frac{1}{\sqrt{5}}(1+ax+a^2x^2+\cdots) - \frac{1}{\sqrt{5}}(1+bx+b^2x^2+\cdots)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(a^n-b^n)x^n$$

$$\Rightarrow \boxed{F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \qquad \forall n.}$$

Ex: Suppose $a_0 = 0$, $a_1 = 2$,

$$a_n = 4a_{n-1} - 4a_{n-2} \qquad (a_2 = 8,\ a_3 = 24, \ldots)$$

$A(x) = \sum a_n x^n$ $\qquad$ char poly: $1 - 4x + 4x^2$

$A(x)(1 - 4x + 4x^2)$:

$$\begin{aligned}
A(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
-4xA(x) &= - 4a_0 x - 4a_1 x^2 - 4a_2 x^3 + \cdots \\
+4x^2 A(x) &= \ \ 4a_0 x^2 + 4a_1 x^3 + \cdots \\
\hline
&= a_0 + (a_1 - 4a_0)x + 0 + 0 + \cdots \\
&= 2x
\end{aligned}$$

$$\Rightarrow A(x) = \frac{2x}{1 - 4x + 4x^2} = \frac{2x}{(1-2x)^2}$$

$\leftarrow$ can't do partial fractions

Instead recall: $\displaystyle \frac{1}{(1-x)^2} = \sum \left(\!\!\binom{2}{k}\!\!\right) x^k = \sum \binom{k+1}{k} x^k = \sum (k+1) x^k$

$$\Rightarrow \frac{2x}{(1-2x)^2} = 2x \sum (k+1) 2^k x^k = \sum (k+1) 2^{k+1} x^{k+1} = \sum n \cdot 2^n x^n$$

$\Rightarrow \boxed{a_n = n \cdot 2^n}$

Thm: For solving linear recurrences:

Suppose $\{a_i\}$ defined by initial values

$a_0, \ldots, a_{d-1}$ and recursion

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d} = 0.$$

Let $p(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_d x^d$. $\leftarrow$ char poly.

Then $A(x)p(x) = q(x)$ for some poly. $q(x)$.

Moreover, let $p(x) = (1 - r_1 x)^{\alpha_1} \cdots (1 - r_k x)^{\alpha_k}$.

Then $\boxed{a_n = \sum_{i=1}^{k} f_{\alpha_i}(n)\, r_i^n}$

where $f_{\alpha_i}(n)$ are polynomials of degree $\leq \alpha_i - 1$.

(Follows from same gen. fn. argument)

Ex: $b_0 = 1$, $b_1 = 3$, $b_2 = 6$,

$$b_n = 3b_{n-1} - 3b_{n-2} + b_{n-3}$$

Infinite products and convergence

Claim: $\displaystyle\sum_{n=0}^{\infty} p(n)x^n = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots$

What does the infinite product mean?

Def: Let $F_1(x), F_2(x), \ldots$ be a sequence of generating functions where

$$F_i(x) = \sum_{j=0}^{\infty} f_{ij}x^j .$$

Let $F(x) = \sum f_n x^n$ be a g.f. as well. Then we say $\{F_i\}$ <u>converges</u> to F, or $\lim_{i\to\infty} F_i(x) = F(x)$, if $\forall j, \exists N$ s.t. $f_{ij} = f_j$ whenever $i > N$.

Ex:

$F_1 = 1$

$F_2 = 1 + x$

$F_3 = 1 + x + x^2$

$\vdots$

$\downarrow$

$F = 1 + x + x^2 + x^3 + \cdots$

Ex: $\displaystyle\prod_{i=0}^{\infty}\left(\frac{1}{1-x^i}\right) = \lim_{n\to\infty}\prod_{i=1}^{n}\frac{1}{1-x^i}$

$\displaystyle = \sum_{n=0}^{\infty} p(n)x^n$