Math 501: Combinatorics
Homework 9

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] Use the Matrix-Tree theorem to verify the computation on page 62 of Enumerative Combinatorics Vol 2, namely that in the graph consisting of a 3×3 lattice grid of vertices, connected by all possible horizontal and vertical edges of length 1, there are exactly 192 spanning trees.

2. (1) [1 point] Does the graph in the previous problem admit an Eulerian tour?

3. (2+) [4 points] The *complete bipartite graph* $K_{m,n}$ is the undirected graph on $[m+n]$ in which there is an edge between i and j if and only if $i \in \{1, 2, \ldots, m\}$ and $j \in \{m+1, \ldots, n\}$ (or vice versa). Show that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.

4. (2) [3 points] Find the number of spanning trees in the *deleted complete graph*: the graph formed by starting with the complete graph K_n and removing the edge connecting vertex 1 and n.

5. (1) [1 point] Calculate the number of spanning trees rooted at a vertex v in the directed cycle graph C_n in two ways: using the matrix-tree theorem, and by simple observation. Check that they match.

6. Let $d > 1$ be a positive integer. A *d-ary de Bruijn sequence* of degree n is a sequence of length d^n containing every length n sequence in $\{0, 1, 2, \ldots, d-1\}^n$ exactly once as a circular factor.

   (a) (1+) [2 points] Show that there always exists a d-ary de Bruijn sequence of degree n for any n.

   (b) (2) [3 points] Find the number of d-ary de Bruijn sequences of degree n that begin with n zeroes.

7. (2-) [3 points] An **undirected Eulerian tour** is a tour on the edges of an undirected graph using every *undirected* edge exactly once (in just one direction). Derive necessary and sufficient conditions for the existence of an undirected Eulerian tour in an undirected graph. Prove your result.

8. The *adjacency matrix* of a directed graph $D = (V, E)$ on vertex set $V = \{1, 2, \ldots, n\}$ is the matrix $A = (a_{ij})$ whose i, j entry is 1 if $(i, j) \in E$ and 0 otherwise.

   (a) (1+) [2 points] Show that the i, j entry of A^k is the number of directed paths of length k from i to j in D.

   (b) (1) [1 point] Verify that the following equality holds, where we consider both sides as formal power series in x with coefficients in the ring of $n \times n$ matrices over $\mathbb{Q}$:

   $$(I - Ax)^{-1} = 1 + Ax + A^2x^2 + A^3x^3 + \cdots$$

   (c) (1+) [2 points] Using the previous part along with the explicit formula for the inverse of a matrix (in terms of the matrix of cofactors), show that the generating function for the number of paths $p_{i,j}(n)$ of length n from vertex i to j is given by

   $$\sum_{n=0}^{\infty} p_{i,j}(n)x^n = \frac{(-1)^{i+j}\det(I - Ax; j, i)}{\det(I - Ax)}$$

   where $\det(I - Ax; j, i)$ is the determinant of the matrix minor formed by removing row j and column i from $I - Ax$. This result is called the **transfer-matrix method**, as it gives a method of proving a sequence has a rational generating function, by showing that the sequence counts paths in a certain directed graph.

(d) (2-) [3 points] Let b_n be the number of sequences of length $n+1$ with entries from $\{1,2,3\}$ that start with 1, end with 3, and do not contain the subsequences 22 or 23. Find a closed formula for the generating function of b_n using the transfer-matrix method, by constructing a directed graph in which certain paths are counted by b_n. You may use a computer to calculate the determinants, but you must write out the directed graph and the corresponding matrices.

9. (2+) [4 points] Prove the (directed) "Matrix-Forest" theorem, which is the following 'q-analog' of the Matrix-Tree theorem. Given a directed graph $D = (V, E)$ on n vertices, define the *augmented Laplacian* to be $L_q(D) = L(D) + qI_n$ where q is a formal variable and I_n is the $n \times n$ identity matrix. Show that

$$\det(L_q(D)) = \sum_{k=1}^{n} F_k(D) q^k$$

where $F_k(D)$ is the number of **oriented spanning forests** of D having k trees, that is, forests that use all vertices in V such that every tree in the forest is an oriented tree towards some root. Thus, in particular, an oriented spanning forest having k trees has k specified roots.

10. (2+) [4 points] Stanley chapter 5 problem 5.66.

11. (5-) [∞ points] Stanley chapter 5 problem 5.73.